Math Diversion 742

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Learning is a treasure that will follow its owner everywhere.
— Chinese proverb

The problem is found at:

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Source: https://www.youtube.com/watch?v=9aEkY3OSsBw
Title:  The Complex Equation That Outsmarts Expectations
     | P 561
 Presenter:  aplusbi
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1 Problem

Given the relation

$$i^{i^z} = i\,, \tag{1}$$

find the complex values of z.

2 Solution

One way to proceed would be to recast (1) as

$$i^{i^z-1} = 1 = e^{2\pi i n} \quad n \in \mathbb{Z}\,. \tag{2}$$

Next, we apply the natural logarithm

$$(i^z - 1)(\ln i) = 2\pi i n \quad n \in \mathbb{Z}\,, \tag{3}$$

or

$$(i^z - 1)(\pi i/2) = 2\pi i n \quad n \in \mathbb{Z}\,. \tag{4}$$

Thus

$$i^z - 1 = \frac{2\pi i n}{\pi i/2} \quad n \in \mathbb{Z}\,, \tag{5}$$

or

$$i^z = 4n + 1 \quad n \in \mathbb{Z}\,. \tag{6}$$

Now we apply the natural logarithm once more

$$z(\pi i/2) = \ln{(4n+1)} \quad n \in \mathbb{Z}\,, \tag{7}$$

or

$$z = \frac{\ln(4n+1)}{(\pi i/2)} \quad n \in \mathbb{Z}, \tag{8}$$

and finally,

$$z = \frac{2}{\pi i} \ln(4n+1) \quad n \in \mathbb{Z}. \tag{9}$$