Math Diversion Problem 746

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He who would ride two camels, finds he can ride neither.
— From an old movie

The material here is found at:

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Source: The Ether of Great Mathematical Ideas
Title: Volume lemma for a reversible process on an ideal gas
 Presenter: Patrick
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1 The Problem

Starting with an ideal gas of fundamental relation

$$PV = nRT\,. \tag{1}$$

show that when taken through a Carnot cycle by reversible processes, as shown in Fig. 1, that

$$\frac{V_2V_4}{V_1V_3} = 1\,. \tag{2}$$

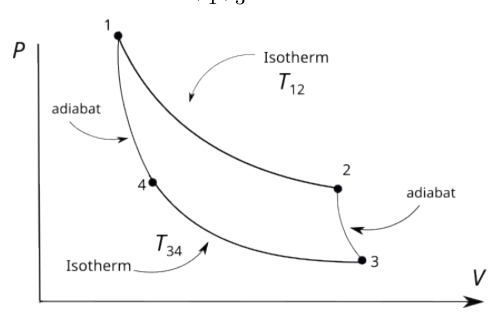

Figure 1. For the path 1 to 2, the ideal gas undergoes reversible isothermal expansion. For 2 to 3, the gas undergoes adiabatic expansion. From 3 to 4, the gas undergoes reversible isothermal compression. And from 4 to 1, the gas undergoes an adiabatic compression.

Although we don't need to know how the heat flows during this cycle, I'll include it for interest's sake. Heat moves into the gas on path segment 1 to 2, and heat flows out of the cycle on path segment 3 to 4. Where does this heat come from and go to? It comes from the surroundings, and returns to it.

Now, for adiabats in this cycle, the governing relation over T and V is given by

$$TV^{\gamma-1} = \text{const}\,, \tag{3}$$

where γ is a constant that we won't need to know.

2 The Solution

Over path 2 to 3, we have that

$$T_2 V_2^{\gamma-1} = \text{const} = T_3 V_3^{\gamma-1}\,, \tag{4}$$

from which we conclude that

$$\frac{T_2}{T_3}\left(\frac{V_2}{V_3}\right)^{\gamma-1} = \frac{T_1}{T_4}\left(\frac{V_1}{V_4}\right)^{\gamma-1}\,. \tag{5}$$

But $T_2 = T_1$ and $T_3 = T_4$, hence this last equation reduces to

$$\left(\frac{V_2}{V_3}\right)^{\gamma-1} = \left(\frac{V_1}{V_4}\right)^{\gamma-1}\,, \tag{6}$$

or

$$\left(\frac{V_2 V_4}{V_1 V_3}\right)^{\gamma-1} = 1\,. \tag{7}$$

Finally, on the assumption that $\gamma - 1 \neq 0$, we have that

$$\frac{V_2 V_4}{V_1 V_3} = 1\,. \tag{8}$$