Math Diversion Problem 750

P. Reany

August 8, 2025

A good short pencil can be better
than a faulty long memory.
— A former teacher of mine

The material here is found at:

```
Source: The Ether of Great Mathematical Ideas
Title: Show that on the Carnot cycle of an ideal
         gas $S$ is a state function.
 Presenter: Patrick
```

1 The Problem

Starting with an ideal gas of fundamental relation

$$PV = nRT\,. \tag{1}$$

show that on the Carnot cycle of an ideal gas S is a state function. In Problem 746, we've already shown that

$$\frac{V_2 V_4}{V_1 V_3} = 1\,. \tag{2}$$

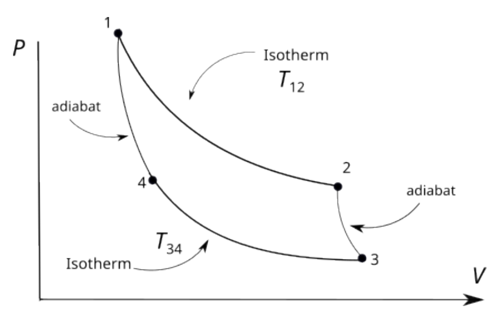

Figure 1. For the path 1 to 2, the ideal gas undergoes reversible isothermal expansion. For 2 to 3, the gas undergoes adiabatic expansion. From 3 to 4, the gas undergoes reversible isothermal compression. And from 4 to 1, the gas undergoes an adiabatic compression.

Heat moves into the gas on path segment 1 to 2, and heat flows out of the cycle on path segment 3 to 4. Where does this heat come from and go to? It comes from the surroundings, and returns to it.

Now, for adiabats in this cycle, no heat is transferred between the substance and the surroundings, so that during those paths, no change in entropy can occur. That means that the total change in entropy occurs only on the isotherm paths.

$$\Delta S = \frac{Q_{12}}{T_{12}} + \frac{Q_{34}}{T_{34}} \,, \tag{3}$$

Useful fact: The internal energy U of an ideal gas is a state function, and is only a function of its temperature T:

$$U = U(T) \,. \tag{4}$$

Technicality: To ensure that this relation will hold on the ideal gas, isothermal expansion and compression must be done slowly enough so that the temperature of the gas is homogeneous throughout. This requires that expansion and compression occur so slowly that the gas is essentially always in thermodynamic 'equilibrium'. I know, it seems somewhat contradictory, but we're forced to live with this paradox to do the analysis.

2 The Solution

Since $\Delta U_{12} = 0$, that implies that[1]

$$Q_{12} = +\int_{V_1}^{V_2} P\,dV = \int_{V_1}^{V_2} \frac{nRT_{12}}{V}\,dV = nRT_{12}\int_{V_1}^{V_2} \frac{1}{V}\,dV = nRT_{12}\ln\frac{V_2}{V_1} \,, \tag{5}$$

where we used the ideal gas law (1). Similarly, since $\Delta U_{34} = 0$, this implies that

$$Q_{34} = \int_{V_3}^{V_4} P\,dV = nRT_{34}\ln\frac{V_4}{V_3} \,. \tag{6}$$

Returning to (3), we have that

$$\begin{aligned}
\Delta S &= \frac{Q_{12}}{T_{12}} + \frac{Q_{34}}{T_{34}} \\
&= nRT_{12}\ln\frac{V_2}{V_1}\frac{1}{T_{12}} + nRT_{34}\ln\frac{V_4}{V_3}\frac{1}{T_{34}} \\
&= nR\left[\ln\frac{V_2}{V_1} + \ln\frac{V_4}{V_3}\right] = nR\left[\ln\frac{V_2V_4}{V_1V_3}\right] \\
&= nR\,[\ln 1] = 0 \,,
\end{aligned} \tag{7}$$

where we used (2).

[1]Temperature is constant throughout.