Math Diversion Problem 752

P. Reany

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> Math is just hard.
> — Alex Wei
> (an OpenAI IMO trainer)

The material here is found at:

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Source: The Ether of Great Mathematical Ideas
Title: A lemma on the Carnot cycle of an ideal gas
 Presenter: Copilot
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1 The Setup

After I had Copilot double check my computations in establishing that for an ideal monoatomic gas in the Carnot cycle, analyzed in the PV plane, that

$$\frac{V_2 V_4}{V_1 V_3} = 1\,, \tag{1}$$

Copilot suggested that I do the same proof in the TS plane for a comparison. I agreed and went to my resources for a reference on it, but I found nothing. So, I asked Copilot to help me do it. What follows is mostly the work of Copilot, though I edited it a bit.

2 The Problem

Let's reconstruct the volume identity

$$\frac{V_2 V_4}{V_1 V_3} = 1 \tag{2}$$

within the Carnot cycle using thermodynamic principles, focusing on reversible processes and the ideal gas law in both PV and TS planes.

Starting Point: Carnot Cycle Process Overview In the PV diagram, the cycle includes:

$1 \to 2$: Isothermal expansion at T_H

$2 \to 3$: Adiabatic expansion

$3 \to 4$: Isothermal compression at T_C

$4 \to 1$: Adiabatic compression

Each leg affects volume and entropy differently depending on temperature and reversibility.

3 The Solution in the PV plane

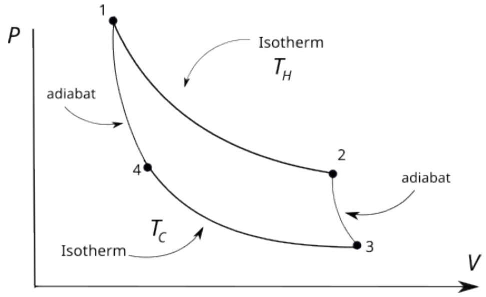

Figure 1. For the path 1 to 2, the ideal gas undergoes reversible isothermal expansion. For 2 to 3, the gas undergoes adiabatic expansion. From 3 to 4, the gas undergoes reversible isothermal compression. And from 4 to 1, the gas undergoes an adiabatic compression.

The ideal gas law

$$PV = nRT \tag{3}$$

applied to an isotherm implies that $PV =$ constant.

During adiabatic processes, $TV^{\gamma-1} =$ constant, for a monoatomic gas where

$$\gamma = \frac{C_P}{C_V} \tag{4}$$

Now let's tie the volume ratios to the adiabatic processes:

From $2 \to 3$: Adiabatic expansion from high temperature T_H to T_C

$$T_H V_2^{\gamma-1} = T_C V_3^{\gamma-1} \tag{5}$$

$\Longrightarrow$

$$\left(\frac{V_2}{V_3}\right)^{\gamma-1} = \frac{T_C}{T_H} \tag{6}$$

From $4 \to 1$: Adiabatic compression from T_C to T_H

$$T_C V_4^{\gamma-1} = T_H V_1^{\gamma-1} \Longrightarrow \left(\frac{V_4}{V_1}\right)^{\gamma-1} = \frac{T_H}{T_C} \tag{7}$$

Multiplying the two ratios:

$$\left(\frac{V_2}{V_3}\right)^{\gamma-1} \cdot \left(\frac{V_4}{V_1}\right)^{\gamma-1} = \frac{T_C}{T_H} \cdot \frac{T_H}{T_C} = 1 \tag{8}$$

Which implies:

$$\left(\frac{V_2}{V_4}\frac{V_1}{V_3}\right)^{\gamma-1} = 1 \longrightarrow \frac{V_2}{V_4}\frac{V_1}{V_3} = 1 \tag{9}$$

4 The Solution in the TS plane

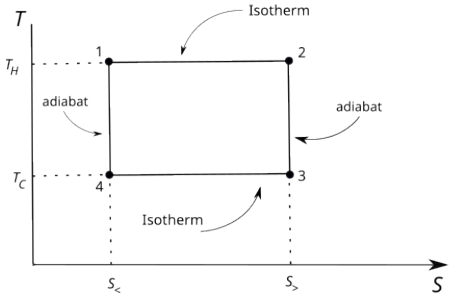

Figure 1. For the path 1 to 2, the ideal gas undergoes reversible isothermal expansion. For 2 to 3, the gas undergoes adiabatic expansion. From 3 to 4, the gas undergoes reversible isothermal compression. And from 4 to 1, the gas undergoes an adiabatic compression.

Although TS diagrams aren't volume-explicit, entropy S changes during isothermal processes are:

For $1 \rightarrow 2$:

$$\Delta S = nR\ln\left(\frac{V_2}{V_1}\right) \tag{10}$$

For $3 \rightarrow 4$:

$$\Delta S = nR\ln\left(\frac{V_4}{V_3}\right) \tag{11}$$

Equal and opposite entropy exchanges across the isothermal legs confirm that:

$$\ln\left(\frac{V_2}{V_1}\right) + \ln\left(\frac{V_4}{V_3}\right) = 0 \longrightarrow \frac{V_2}{V_1} \cdot \frac{V_4}{V_3} = 1 \longrightarrow \frac{V_2 V_4}{V_1 V_3} = 1\,. \tag{12}$$