Math Diversion Problem 755: The Pauli Algebra, Part 1

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A conversation between a father and son:

"Can I have a motorcycle when I get old enough?"
"If you take care of it."
"What do you have to do?"
"Lot's of things. You've been watching me."
"Will you show me all of them?"
"Sure."
"Is it hard?"
"Not if you have the right attitudes. It's having the right attitudes that's hard."
— Robert Pirsig to his son, from *Zen and the Art of Motorcycle Maintenance*

The material here is found at:

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Source: The Ether of Great Mathematical Ideas
Title: An Identity in the Pauli Algebra
 Presenter: Patrick
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1 The Series Introduction

This article begins a series on the Pauli algebra within the Pauli theory of electron spin. Each article will likely be fairly short and deal with just one topic. I'll introduce little bits of the Pauli algebra as the series progresses. I'm a strong believer that in a series like this one, that there should be no technical information core dump at the beginning (though that may well be appropriate for a book). I follow a parallel to the rule in fiction: Don't introduce too many characters in the first chapter.

2 The Problem

In quantum mechanics of electron spin, we have the formalism known as Pauli Theory, and within it, the Pauli Algebra. Within this algebra is the following identity, which we are to prove.[1]

$$(\mathbf{a}\cdot\boldsymbol{\sigma})(\mathbf{b}\cdot\boldsymbol{\sigma}) = \mathbf{a}\cdot\mathbf{b}\mathbf{I} + i\mathbf{a}\times\mathbf{b}\cdot\boldsymbol{\sigma}\,, \tag{1}$$

where the sigmas are the Pauli matrices and **a** and **b** are vectors, and $\mathbf{a}\cdot\mathbf{b}$ is the usual Gibbs' inner product of vectors **a** and **b**. The meaning of the $\boldsymbol{\sigma}$ symbol will be given next.

3 The Preparation

In the Pauli algebra, Eq. (1) is technically a matrix equation, though we will not be using matrix operations, per se. Instead, we will be using the formal properties of the Pauli matrices, which we'll see in a moment.

To begin with, the Pauli matrices are as follows,

$$\boldsymbol{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

These three matrices plus the **identity matrix**

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{3}$$

form the basis form a 4-D complex linear space. The algebra constructed on these four matrices is called the **Pauli algebra**.[2]

Formal Properties

The simplest way to describe the 'formal' properties I'm interested in is to present the Pauli matrices in the following form:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{4}$$

where we have made the well-accepted correspondences

$$x \to 1, \quad y \to 2, \quad z \to 3\,. \tag{5}$$

[1] As I could not find a name for this identity, I asked Copilot for some established name in the literature. From its list provided of possible names, I decided on the **Vector Pauli Identity**.

[2] In the Pauli algebra, matrices can multiply each other in the usual way, and they can be scalar multiplied by complex numbers. The resulting algebra is a **matrix representation** of the Pauli algebra.

Then,

$$\boldsymbol{\sigma}_i \boldsymbol{\sigma}_j = -\boldsymbol{\sigma}_j \boldsymbol{\sigma}_i \quad \text{for } i, j \in [1, 2, 3]\,. \tag{6}$$

The last formal properties of the Pauli matrices that we'll need are these products:

$$\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 = i\boldsymbol{\sigma}_3, \quad \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_1, \quad \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_3 = -i\boldsymbol{\sigma}_2\,, \tag{7a}$$

$$\boldsymbol{\sigma}_i^2 = \mathbf{I} \quad \text{for } i \in [1, 2, 3]\,, \tag{7b}$$

$$\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = i\mathbf{I}\,. \tag{7c}$$

These last four equations are inter-related. For example, the relations in (7a) can be determined from the other three. In particular, starting with (7c) and multiplying on the left by $\boldsymbol{\sigma}_1$, we get

$$\boldsymbol{\sigma}_1^2 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = \boldsymbol{\sigma}_1 i\mathbf{I}\,. \tag{8}$$

But since $\boldsymbol{\sigma}_1^2 = \mathbf{I}$ and $\boldsymbol{\sigma}_1 i\mathbf{I} = i\boldsymbol{\sigma}_1$, we have that

$$\boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = i\boldsymbol{\sigma}_1\,. \tag{9}$$

To get $\boldsymbol{\sigma}_1 \boldsymbol{\sigma}_3 = -i\boldsymbol{\sigma}_2$ start with

$$i\mathbf{I} = \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_2 \boldsymbol{\sigma}_3 = -\boldsymbol{\sigma}_2 \boldsymbol{\sigma}_1 \boldsymbol{\sigma}_3 \tag{10}$$

and continue from there.

What does $\mathbf{a} \cdot \boldsymbol{\sigma}$ mean?

It's just a shorthand for the following

$$\mathbf{a} \cdot \boldsymbol{\sigma} = a_1 \boldsymbol{\sigma}_1 + a_2 \boldsymbol{\sigma}_2 + a_3 \boldsymbol{\sigma}_3\,, \tag{11}$$

where $a_i = \mathbf{a} \cdot \boldsymbol{\sigma}'_i$, and where $\boldsymbol{\sigma}'_i$ is the usual Gibbs' unit basis vector for each $i \in [1, 2, 3]$, which correspond to unit vectors in the x, y, z directions. Hence, we can express the vector $\mathbf{a}$ as

$$\mathbf{a} = a_1 \boldsymbol{\sigma}'_1 + a_2 \boldsymbol{\sigma}'_2 + a_3 \boldsymbol{\sigma}'_3\,. \tag{12}$$

And while we're at it,

$$(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma} = (\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}'_1 \boldsymbol{\sigma}_1 + (\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}'_2 \boldsymbol{\sigma}_2 + (\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}'_3 \boldsymbol{\sigma}_3\,. \tag{13}$$

Or if you're more accustomed to seeing standard unit basis vectors in the form of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ then (13) becomes

$$(\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma} = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_1 \boldsymbol{\sigma}_1 + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_2 \boldsymbol{\sigma}_2 + (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_3 \boldsymbol{\sigma}_3\,. \tag{14}$$

I know, it's a confusing hodgepodge of Gibbs' vectors with Pauli matrices mixed together, and we won't be able to make good sense of it until we redo the theory in the geometric algebra of Hestenes. Until then, we'll just have to deal with it. By the way, keep in mind that in either notation,

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{e}_i \quad \text{or} \quad (\mathbf{a} \times \mathbf{b}) \cdot \boldsymbol{\sigma}'_i \quad \text{for} \quad i \in [1, 2, 3] \tag{15}$$

is just a scalar in the complex numbers. And, by the way, in the $\mathbf{e}_i$ basis, (12) becomes

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 \,. \tag{16}$$

4 The Solution

Now we're read to begin

$$\begin{aligned}
(\mathbf{a} \cdot \boldsymbol{\sigma})(\mathbf{b} \cdot \boldsymbol{\sigma}) &= (a_1\boldsymbol{\sigma}_1 + a_2\boldsymbol{\sigma}_2 + a_3\boldsymbol{\sigma}_3)(b_1\boldsymbol{\sigma}_1 + b_2\boldsymbol{\sigma}_2 + b_3\boldsymbol{\sigma}_3) \\
&= (a_1b_1 + a_2b_2 + a_3b_3)\mathbf{I} + (a_1b_2 - b_1a_2)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_2 \\
&\qquad + (a_2b_3 - b_2a_3)\boldsymbol{\sigma}_2\boldsymbol{\sigma}_3 + (a_1b_3 - b_3a_3)\boldsymbol{\sigma}_1\boldsymbol{\sigma}_3 \\
&= \mathbf{a} \cdot \mathbf{b}\mathbf{I} + i(a_1b_2 - b_1a_2)\boldsymbol{\sigma}_3 \\
&\qquad + i(a_2b_3 - b_2a_3)\boldsymbol{\sigma}_1 - i(a_1b_3 - b_3a_3)\boldsymbol{\sigma}_2 \\
&= \mathbf{a} \cdot \mathbf{b}\mathbf{I} + i\mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma}'_3\boldsymbol{\sigma}_3 + i\mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma}'_1\boldsymbol{\sigma}_1 + i\mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma}'_2\boldsymbol{\sigma}_2 \\
&= \mathbf{a} \cdot \mathbf{b}\mathbf{I} + i\big(\mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma}'_1\boldsymbol{\sigma}_1 + \mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma}'_2\boldsymbol{\sigma}_2 + \mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma}'_3\boldsymbol{\sigma}_3\big) \\
&= \mathbf{a} \cdot \mathbf{b}\mathbf{I} + i\mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma} \,.
\end{aligned} \tag{17}$$

5 Sample Calculation

Let's do a sample calculation. Generally,

$$\mathbf{a} \times \mathbf{b} \cdot \mathbf{c} = \mathbf{c} \cdot \mathbf{a} \times \mathbf{b} = \det \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \,, \tag{18}$$

therefore, specifically,

$$\mathbf{a} \times \mathbf{b} \cdot \boldsymbol{\sigma}'_1 = \boldsymbol{\sigma}'_1 \cdot \mathbf{a} \times \mathbf{b} = \det \begin{vmatrix} 1 & 0 & 0 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = a_2b_3 - b_2a_3 \,. \tag{19}$$