

Math Diversion Problem 756

P. Reany

August 13, 2025

In life, work hard, don't complain, emphasize the
positive, earn your accomplishments.
— The Author

The material here is found at:

Source: https://www.youtube.com/watch?v=B26vyP_917k
Title: A Nice Algebra Problem
Presenter: SALogic

1 The Preface

I did not view the Presenter's solution, nor the solutions posted in the comments, except to see that they got the integer 4 for their answers, but which I was not clever enough to get.

2 The Problem

Given the relation

$$\log_{\frac{3}{4}}\left(\frac{x}{3}\right) + \log_{\frac{1}{2}}\left(\frac{x}{2}\right) = -2, \quad (1)$$

find the real values of x .

3 The Solution

There won't always be a slick solution to logarithmic problems that yields an integer, hence it's worth seeing how to work with real numbers on this problem.

Let's begin by raising $1/2$ to the last equation:

$$\left(\frac{1}{2}\right)^{\log_{\frac{3}{4}}\left(\frac{x}{3}\right)} \left(\frac{x}{2}\right) = \left(\frac{1}{2}\right)^{-2} = 4, \quad (2)$$

But

$$\log_{\frac{3}{4}}\left(\frac{x}{3}\right) = \frac{\log_{\frac{1}{2}}\left(\frac{x}{3}\right)}{\log_{\frac{1}{2}}\left(\frac{3}{4}\right)}. \quad (3)$$

Let's call

$$\alpha = \frac{1}{\log_{\frac{1}{2}}\left(\frac{3}{4}\right)} \approx 2.40940, \quad (4)$$

then (3) becomes

$$\log_{\frac{3}{4}}\left(\frac{x}{3}\right) = \alpha \log_{\frac{1}{2}}\left(\frac{x}{3}\right) = \log_{\frac{1}{2}}\left(\frac{x}{3}\right)^\alpha. \quad (5)$$

Plugging this result into (2), we have that

$$\left(\frac{x}{3}\right)^\alpha \left(\frac{x}{2}\right) = 4, \quad (6)$$

which simplifies to

$$x = (8 \times 3^\alpha)^{1/(\alpha+1)}. \quad (7)$$

So, plugging in the approximate value we obtained for α above, we get

$$x \approx 4.0007. \quad (8)$$

This value is close enough to 4 to suggest that it's the actual solution.