Math Diversion Problem 763

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> You cannot ask us to take sides against arithmetic.
> — Winston Churchill

The material here is found at:

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Source: https://www.youtube.com/watch?v=ySvzli6KhKo
Title: Japanese || Olympiad Logarithmic Math problem |
 Presenter: Mathpoints
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1 Introduction

Our first task will be to connect the familiar construction of the ellipse on a flat surface to its standard equation in the x, y-plane. In the last section, we will present the standard names parts of the ellipse.

2 Theorem One

It's a well-known trick that one can draw an ellipse by affixing a length of string to two fixed points in a plane. The length of this string must be longer than the distance between the two fixed points. Then place a sharpened pencil against the string and push the pencil against the string with just enough force to keep the string taut and then slide the pencil tip freely left and right over the flat surface, while maintaining the same tautness of the pencil against the string.

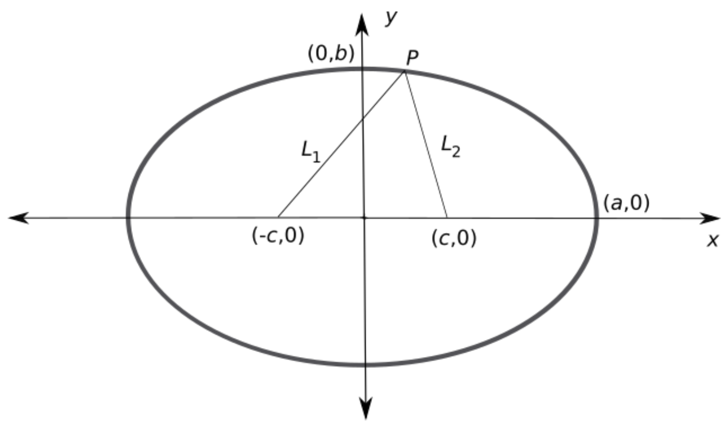

Figure 1. The ends of a string are attached to a flat surface at points $(-c, 0)$ and $(c, 0)$ shown as lying on the x-axis. The points at which the curve crosses the x-axis are at $(a, 0)$ and $(-a, 0)$. It crosses the y-axis at $(0, b)$ and $(0, -b)$ Of course, the origin of the coordinate system lies halfway between the fixed points, $(-c, 0)$ and $(c, 0)$, which are known as the *foci* of the ellipse.

If we assume that the string is inextensible during this operation, then the total length of the string is constant (say, equal to L) and this is required for our analysis.

We're defining the curve that can be constructed by the above procedure as an *ellipse*.

Now, our first major theorem is to how that

$$c^2 = a^2 - b^2 \,. \tag{1}$$

3 Proof of Theorem One

By our construction

$$L = L_1 + L_2 \,, \tag{2}$$

where L_1 and L_2 are the lengths of the string segments from the point P to the fixed points $(-c, 0)$ and $(c, 0)$. Of course, the values in (1), although algebraically precise, are, in practice, only approximately precise because of the finite thickness of the strng and the pencil tip and related issues of its construction on paper.

Think of the point P in Fig. 1 as where the tip of the pencil is at a given moment. When P is on the y-axis, then $L_1 = L_2$ and we can used the Pythagorean Formula to prove that

$$L_1^2 = b^2 + (-c)^2 \quad \text{and} \quad L_2^2 = b^2 + c^2 \,. \tag{3}$$

So, we're halfway there. All we need now is a relationship between the L's and the number a. When the point P lines on the $+x$-axis at point $(a, 0)$, then

$L_1 = |\, a - (-c)\, | = a + c$ and $L_2 = |\, a - (c)\, | = a - c$. Thus,

$$L = L_1 + L_2 = a + c + a - c = 2a\,. \tag{4}$$

But from (3) we also have that

$$L = L_1 + L_2 = 2\sqrt{b^2 + c^2}\,. \tag{5}$$

On combining these last two equations, we have that

$$a = \sqrt{b^2 + c^2}\,. \tag{6}$$

We can solve this last equation for c^2 to get

$$c^2 = a^2 - b^2\,. \tag{7}$$

4 Theorem Two

Show that the equation for the ellipse can be expressed in coordinates as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{8}$$

5 Proof of Theorem Two

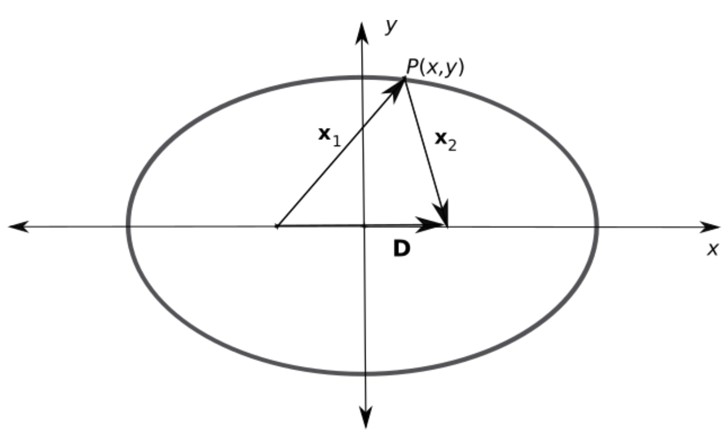

Figure 2. Here, we setup the problem in terms of abstract vectors.

We begin with the equation

$$\mathbf{D} = \mathbf{x}_1 + \mathbf{x}_2\,, \tag{9}$$

and, as before, we add the constraint,

$$|\, \mathbf{x}_1 \,| + |\, \mathbf{x}_2 \,| = \rho\,, \tag{10}$$

where, in time, we'll substitute that $\rho = 2a$. Let the vector $\mathbf{x}$ be the vector going from the origin to the point $P = (x, y)$. Then

$$\mathbf{x} = \tfrac{1}{2}\mathbf{D} - \mathbf{x}_2 = +\mathbf{x}_1 - \tfrac{1}{2}\mathbf{D}\,, \tag{11}$$

Now let's write $\mathbf{x}_1$ and $\mathbf{x}_2$ in terms of coordinates:

$$\mathbf{x}_1 = (x + \tfrac{1}{2}D, y)\,, \quad \mathbf{x}_2 = (-x + \tfrac{1}{2}D, -y)\,, \tag{12}$$

But, since $\frac{1}{2}D = c$, we have

$$\mathbf{x}_1 = (x + c, y)\,, \quad \mathbf{x}_2 = (-x + c, -y)\,, \tag{13}$$

On squaring (10) and then rearranging it, we get that

$$2\,|\,\mathbf{x}_1\,|\;|\,\mathbf{x}_2\,| = \rho^2 - (|\,\mathbf{x}_1\,|^2 + |\,\mathbf{x}_2\,|^2)\,, \tag{14}$$

Squaring this last equation gives

$$4\,|\,\mathbf{x}_1\,|^2\,|\,\mathbf{x}_2\,|^2 = \rho^4 - 2\rho^2(|\,\mathbf{x}_1\,|^2 + |\,\mathbf{x}_2\,|^2) + |\,\mathbf{x}_1\,|^4 + |\,\mathbf{x}_2\,|^4 + 2\,|\,\mathbf{x}_1\,|^2\,|\,\mathbf{x}_2\,|^2\,. \tag{15}$$

Simplifying a bit, we get

$$\begin{aligned} 0 &= \rho^4 - 2\rho^2(|\,\mathbf{x}_1\,|^2 + |\,\mathbf{x}_2\,|^2) + |\,\mathbf{x}_1\,|^4 + |\,\mathbf{x}_2\,|^4 - 2\,|\,\mathbf{x}_1\,|^2\,|\,\mathbf{x}_2\,|^2 \\ &= \rho^4 - 2\rho^2(|\,\mathbf{x}_1\,|^2 + |\,\mathbf{x}_2\,|^2) + (|\,\mathbf{x}_1\,|^2 - |\,\mathbf{x}_2\,|^2)^2\,. \end{aligned} \tag{16}$$

If we add $4\rho^2\,|\,\mathbf{x}_2\,|^2$ to both sides, we get

$$\begin{aligned} 4\rho^2\,|\,\mathbf{x}_2\,|^2 &= \rho^4 - 2\rho^2(|\,\mathbf{x}_1\,|^2 - |\,\mathbf{x}_2\,|^2) + (|\,\mathbf{x}_1\,|^2 - |\,\mathbf{x}_2\,|^2)^2 \\ &= [\,\rho^2 - (|\,\mathbf{x}_1\,|^2 - |\,\mathbf{x}_2\,|^2)\,]^2\,. \end{aligned} \tag{17}$$

From (13), we get

$$|\,\mathbf{x}_1\,|^2 = x^2 + y^2 + 2cx + c^2\,, \tag{18a}$$

$$|\,\mathbf{x}_2\,|^2 = x^2 + y^2 - 2cx + c^2\,. \tag{18b}$$

Hence,

$$|\,\mathbf{x}_1\,|^2 - |\,\mathbf{x}_2\,|^2 = 4cx\,, \tag{19}$$

and since $\rho = 2a$, then (17) becomes

$$16a^2\,|\,\mathbf{x}_2\,|^2 = [\,4a^2 - 4cx\,]^2\,. \tag{20}$$

Simplifying,

$$(a^2 - c^2)x^2 + a^2y^2 = a^4 - a^2c^2\,. \tag{21}$$

Dividing both side by $a^4 - a^2c^2$, we get

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1\,. \tag{22}$$

But from (1) we know that $a^2 - c^2 = b^2$, thus

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,, \tag{23}$$

which is what we were to show.

6 Standard Named Parts of the Ellipse

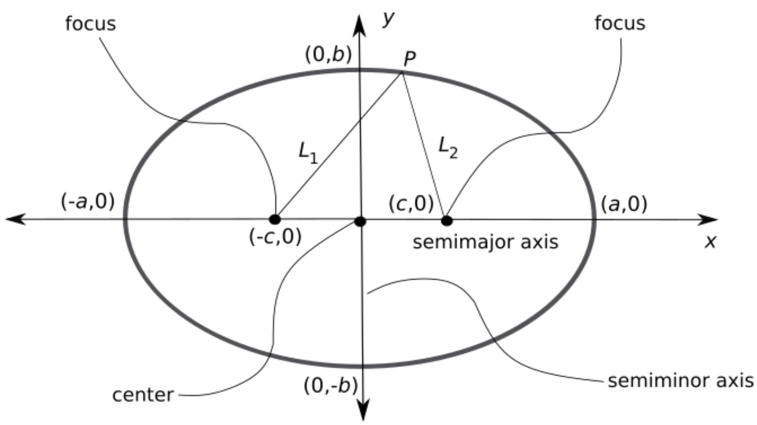

Figure 3. The standard names of the ellipse.

The line segment from points $(-a, 0)$ to $(a, 0)$ is called the *major axis*, which, by the way, is the longer of the two axes. The line segments formed by dividing the major axis at the center point are called the *semimajor axes*. In the y-direction, we also have by similar definition, the *minor axis* and the two *semiminor axes*.

Definition: The *focal length* of an ellipse is the distance from the center of the ellipse to one of its foci. According to Figure 3, that would be the number c.

Definition: The points $(-a, 0)$ and $(a, 0)$ are called the *vertices* of the ellipse.

Definition: The *eccentricity*, denoted as e, is the ratio of the focal length, c, to the length of the semimajor axis, a:

$$e \equiv \frac{c}{a}\,. \tag{24}$$

So, there are two facts worth remembering about eccentricity. First, if we rescale a given ellipse, its eccentricity will remain the same. Second, if we deform an ellipse such that its eccentricity goes to zero, then the shape of the ellipse approaches the shape of a circle.