Math Diversion Problem 764

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> I love it when a plan comes together.
> — Hannibal Smith, *The A-Team*

The material here is found at:

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Source: https://www.youtube.com/watch?v=oEtj2FhRB24
Title: Japanese | Math Olympiad Problem With Natural Log
 Presenter: MathProwess
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1 The Problem

Given the relation

$$-x^2 = 5 \ln x \,, \tag{1}$$

find the values of x.

2 Solution

This looks like a job for the Lambert W function! Let's convert what we've been given into a form on which we can apply a Lambert lemma.

First, let's do a few merely algebraic steps, involving logarithms, to get

$$\frac{1}{5} = x^{-2} \ln x^{-1} \,. \tag{2}$$

Then multiply through by 2 and flip sides:

$$x^{-2} \ln x^{-2} = \frac{2}{5} \,. \tag{3}$$

Next, take the Lambert W function across that last equation, to get

$$\ln x^{-2} = W_n(2/5) \,. \tag{4}$$

From this we get, provisionally,

$$x = \pm e^{-\frac{1}{2} W_n(2/5)} \,. \tag{5}$$

First, we discard the minus sign (since it won't work in (1)), to get

$$x = e^{-\frac{1}{2}W_n(2/5)} \,. \tag{6}$$

WolframAlpha tells me that the only legitimate n values are $\{-1, 0, 1\}$.

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B \,, \tag{7}$$

then

$$z = W(B) \,, \tag{8}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{9}$$

then

$$\ln y = W(y \ln y) = W(B) \,. \tag{10}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A \,, \tag{11}$$

which looks very similar to (7). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s} \,. \tag{12}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A) \,, \tag{13}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)} \,. \tag{14}$$

[1] This notation I invented myself.