Math Diversion Problem 771:
Motivated Solution to the Cubic

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Motivating examples are of the Devil.
— Fredric Schuller

Generally speaking, a *motivated solution* is an attempt by an author or lecturer to present to his or her audience not only the steps to a solution, but also the following three cognetics: 1) a brief description of the particular results being sought, 2) a brief description of the path the solution will follow, 3) an explanation of why the particular path to the the goal rather than some other path.

There are a number of ways to arrive at the roots of the general cubic equation; the one we will take is McKelvey's [1] presentation of the Cardan solution. Specifically, we desire a formula for the roots to the cubic equation similar to the formula for the roots to the quadratic equation. Our starting point will be the general cubic equation

$$z^3 + az^2 + bz + c = 0\,. \tag{1}$$

where a, b, c are real numbers. It's a fairly safe bet that the original path taken from here was the result of many hours trying tricky substitutions and the like, until the right combination was found. The kernel of our method is to employ two variable substitutions to reduce (1) to a simpler form, in particular, to a quadratic form, which we already know how to deal with. Our goal here is not necessarily to follow the historic approach, but rather to explain as best as is possible why the tricky steps are actually good rational choices.

From our experience with integration we know that well-chosen variable substitutions often transform the original integral into an easier integral. For similar reasons we will attempt variable substitutions to simplify the cubic equation. It is well know from the theory of algebraic equations that the substitution

$$z = x - a/3\,. \tag{2}$$

will transform (1) into the following 'reduced' cubic, lacking a quadratic term:

$$x^3 \pm 3\alpha x + 2\beta = 0\,. \tag{3}$$

where

$$\pm 3\alpha = b - a^2/3 \qquad \text{and} \qquad \beta = 2a^3/27 - ab/3 + c \tag{4}$$

We have used the $\pm$ sign in (3) to "factor-out" the sign, making α strictly positive for algebraic convenience later.

The next substitution, namely

$$x = u + v \tag{5}$$

was surely a shot in the dark for whoever tried it first.[1]

This substitution gives us a single equation in two unknowns—which is not solvable! It may at first appear that we have only made matters worse, but this is not the case. By introducing another variable into the equation, although in one sense we have added complications to it, we have also added opportunities to constrain variables u and v to our own liking to produce a coupled system of equations in u and v.

I said we needed to constrain the equation in u and v because we must now solve for u and v, and to do this we must add another equation in u and v to make two equations. This entire approach is predicated on the general principle that n simultaneous equations are needed to solve for a system of n coupled equations.

Now, using (5) in (3) we have

$$u^3 + v^3 + 3(u+v)(uv \pm \alpha) + 2\beta = 0\,. \tag{6}$$

Remember that we have already decided that it will be beneficial to mold the original equation so as to obtain an alternative equation to solve that is a quadratic, because we already know how to solve a quadratic. This quadratic could be in u or u^2 or even u^3 (the variable v enters symmetrically to u so we can deal with one without loss of generality).

If (6) represents in our way of looking at it a constraint $f(u,v)$ on u and v, where $f(u,v) = u^3 + v^3 + 3(u+v)(uv \pm \alpha) + 2\beta = 0$, then a simple and simplifying second constraint could be $uv \pm \alpha = 0$. The advantage of this constraint is obvious, for it not only gives us a simple relation between u and v, namely

$$v = \mp \alpha/u\,, \tag{7}$$

it also simplifies (6) to

$$u^3 + v^3 + 2\beta = 0\,. \tag{8}$$

Now we have two coupled equations in two unknowns, rather than one equation in two unknowns. Substituting v from (7) into (6) yields

$$u^6 + 2\beta u^3 \mp \alpha^3 = 0 \tag{9}$$

[1] Author Urs Oswald [2] credits Italian mathematician Tartaglia as the person who first invented this clever substitution.

which is a quadratic in u^3. Therefore we obtain

$$u^3 = -\beta + \sqrt{\beta^2 \pm \alpha^3} \tag{10a}$$

$$v^3 = \mp\alpha^3/u^3 = \beta - \sqrt{\beta^2 \pm \alpha^3} \tag{10b}$$

From here on, the procedure is straightforward: we back substitute the variables until we arrive at z. It's important to note that when we take the cube roots of and u^3 v^3, we introduce the three cube roots of unity

$$1, \qquad \tfrac{1}{2}(-1 + i\sqrt{3}), \qquad \tfrac{1}{2}(-1 - i\sqrt{3})\,. \tag{11}$$

If we let

$$u_1 = [-\beta + \sqrt{\beta^2 \pm \alpha^3}\,]^{1/3}$$

$$v_1 = [-\beta - \sqrt{\beta^2 \pm \alpha^3}\,]^{1/3} \tag{12a}$$

$$u_2 = \tfrac{1}{2}(-1 + i\sqrt{3})u_1$$

$$v_2 = \tfrac{1}{2}(-1 - i\sqrt{3})v_1 \tag{12b}$$

$$u_3 = \tfrac{1}{2}(-1 - i\sqrt{3})u_1$$

$$v_3 = \tfrac{1}{2}(-1 + i\sqrt{3})v_1 \tag{12c}$$

then we obtain the three roots to (3), namely

$$x_i = u_i + v_i \qquad (i = 1, 2, 3)\,. \tag{13}$$

And the three roots to (1) follow trivially by using (2)

$$z_i = u_i + v_i - a/3 \qquad (i = 1, 2, 3, . \tag{14}$$

From here on, McKelvey re-expresses these roots in terms of trigonometric and hyperbolic forms, which we will not pursue here.

References

[1] J.P. McKelvey, *Simple transcendental expressions for the roots of cubic equations*, *Am. J. Phys.* **52**, 269–270 (1984).

[2] U. Oswald, *The Cubic Equation*, http://www.ursoswald.ch/download/CUBIC.pdf, p. 1 (January 2009).