Math Diversion Problem 775

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In the middle of difficulty lies opportunity.
— John Archibald Wheeler

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Source: https://www.youtube.com/watch?v=NGYX6gtObec
Title:  Introduction to conformal field theory, Lecture 1
 Presenter: Tobias Osborne
 (Read-along notes and a problem to solve.)
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1 Problem

Given the relation (in a conformal field theory)

$$x'^{\mu} = x^{\mu} + \epsilon^{\mu}\,, \tag{1}$$

where ϵ^{μ} is small, show that

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial\epsilon)\eta_{\mu\nu}\,. \tag{2}$$

So what's going on here? We'll define a conformal Lie group acting on a manifold, and then convert to its Lie algebra with the transformation (1). The ShowThat equation (2) is the confluence of what defines a conformal group with (1) and the definition of what defines a conformal group, Eq. (8) below. All this will be set up in the next section and solved in the section after that.

2 Set Up for conformal field theory in d dimensions

First, we define a manifold to work on. Let our manifold M be $\mathbb{R}^{(p,q)}$, where p is the number of basis vectors that square positive, and q is the number of basis vectors that square negative.

Now for the metric:

$$\eta_{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(\underbrace{1,1,\ldots,1}_{p},\underbrace{-1,-1,\ldots,-1}_{q})\,, \tag{3}$$

where

$$p + q \equiv d \quad \text{and} \quad p, q \in \mathbb{Z}^+ \,. \tag{4}$$

Note

$$g^{\mu\nu} g_{\mu\nu} = \delta^\mu_\mu = d \,. \tag{5}$$

Now, let us consider a smooth change of coordinates

$$x \mapsto x'(x) \,, \tag{6}$$

where

$$x = (x^1, x^2, \dots, x^p, x^{p+1}, \dots, x^{p+q}) \,, \tag{7}$$

Now, what defines our conformal group is characterized by smooth transformation that only rescale vectors, so that

$$g'_{\mu\nu} = \Omega(x) \, g_{\mu\nu} \,, \tag{8}$$

where $\Omega(x) > 0$. If we consider the angle $\angle\theta$ between vectors v and u, we have that

$$\angle\theta \equiv \frac{g_{\mu\nu} u^\mu v^\nu}{\sqrt{(g_{\mu\nu} u^\mu v^\nu)^2}} \,, \tag{9}$$

Using (8), we have that

$$\angle\theta' = \frac{g'_{\mu\nu} u^\mu v^\nu}{\sqrt{(g'_{\mu\nu} u^\mu v^\nu)^2}} \tag{10}$$

$$= \frac{\Omega g_{\mu\nu} u^\mu v^\nu}{\sqrt{(\Omega g_{\mu\nu} u^\mu v^\nu)^2}} \tag{11}$$

$$= \frac{g_{\mu\nu} u^\mu v^\nu}{\sqrt{(g_{\mu\nu} u^\mu v^\nu)^2}} \tag{12}$$

$$= \angle\theta \,. \tag{13}$$

Now, generally speaking, g transforms as a rank 2 tensor with transformation

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} \, g_{\alpha\beta} \,, \tag{14}$$

3 Solution

Let's rewrite (1) to

$$x^\alpha = x'^\alpha - \epsilon^\alpha \,, \tag{15}$$

and take the partial derivative of this by x'^μ:

$$\frac{\partial x^\alpha}{\partial x'^\mu} = \delta^\alpha_\mu - \frac{\partial \epsilon^\alpha}{\partial x'^\mu} \,, \tag{16}$$

Hence, (14) becomes

$$g'_{\mu\nu}(x') = \left(\delta^{\alpha}_{\mu} - \frac{\partial\epsilon^{\alpha}}{\partial x'^{\mu}}\right)\left(\delta^{\beta}_{\nu} - \frac{\partial\epsilon^{\beta}}{\partial x'^{\nu}}\right) g_{\alpha\beta}\,, \tag{17}$$

Now, expanding this to first-order in ϵ we have that

$$\begin{aligned} g'_{\mu\nu}(x') &= g_{\mu\nu} - \left(\frac{\partial\epsilon^{\alpha}}{\partial x'^{\mu}}\delta^{\beta}_{\nu} + \frac{\partial\epsilon^{\beta}}{\partial x'^{\nu}}\delta^{\alpha}_{\mu}\right) g_{\alpha\beta} && \text{(18a)} \\ &= g_{\mu\nu} - \left(\frac{\partial\epsilon^{\alpha}}{\partial x'^{\mu}}\delta_{\nu\alpha} + \frac{\partial\epsilon^{\beta}}{\partial x'^{\nu}}\delta_{\mu\beta}\right) && \text{(18b)} \\ &= g_{\mu\nu} - \left(\frac{\partial\epsilon_{\nu}}{\partial x'^{\mu}} + \frac{\partial\epsilon_{\mu}}{\partial x'^{\nu}}\right) && \text{(18c)} \\ &= g_{\mu\nu} - (\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu})\,. && \text{(18d)} \end{aligned}$$

To conform to (8), it's necessary that[1]

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \lambda g_{\mu\nu}\,, \tag{19}$$

where λ is a constant to be determined, which we'll do next. Let's multiply this last equation through by $g^{\mu\nu}$ and sum, to get

$$(\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu})g^{\mu\nu} = \lambda g_{\mu\nu}g^{\mu\nu} = \lambda d\,. \tag{20}$$

The LHS becomes

$$2\partial_{\mu}\epsilon^{\mu} \equiv 2\partial\epsilon\,. \tag{21}$$

Therefore,

$$\lambda = \frac{2}{d}(\partial\epsilon)\,. \tag{22}$$

Going back to (19),

$$\partial_{\mu}\epsilon_{\nu} + \partial_{\nu}\epsilon_{\mu} = \frac{2}{d}(\partial\epsilon)g_{\mu\nu} = \frac{2}{d}(\partial\epsilon)\eta_{\mu\nu}\,. \tag{23}$$

Returning to (18d), I get

$$g'_{\mu\nu} = g_{\mu\nu} - \frac{2}{d}(\partial\epsilon)\eta_{\mu\nu}\,, \tag{24}$$

which, apparently, contains a sign error.

Anyway, we get that

$$\Omega(x) = 1 - \frac{2}{d}(\partial\epsilon)\,. \tag{25}$$

We note that Osborne got a plus sign instead of my minus sign. Finally, we note that for $d = 2$, we might have a special case to deal with.

[1] That is, both terms must be proportional to $g_{\mu\nu}$.