Math Diversion Problem 777

P. Reany

August 29, 2025

I love it when a plan comes together.
— Hannibal Smith, *The A-Team*

```
Source: The Ether of great mathematical ideas
Title: The Area of an Ellipse
 Presenter: Patrick
```

1 Theorem

The area of an ellipse of semimajor length a and semiminor length b is

$$\text{area} = \pi ab\,. \tag{1}$$

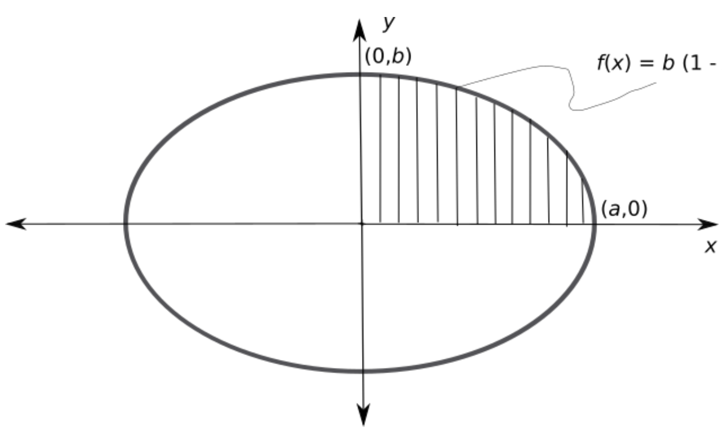

Figure 1. The area of the ellipse is four times the area of its first quadrant contribution.

The standard equation for the ellipse in terms of coordinates is given as

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\,. \tag{2}$$

But this equation is not in the form of a function that we can use to perform a Riemann integration. To fix that, we solve Equation (2) for $y = f(x)$ only in the first quadrant, to get

$$f(x) = b\sqrt{1 - x^2/a^2}\,. \tag{3}$$

Then, performing the integration in the first quadrant, we have that

$$\text{area}_1 = b\int_0^a \sqrt{1 - x^2/a^2}\,dx\,, \tag{4}$$

where area_1 is the area only in the first quadrant.

Making the variable substitution $z = x/a$, the integral becomes

$$\text{area}_1 = ab\int_0^1 \sqrt{1 - z^2}\,dz = \tfrac{1}{4}\pi ab\,. \tag{5}$$

The area of the entire ellipse is then

$$\text{area} = \pi ab\,. \tag{6}$$

I leave it to the interested reader to perform the above integral. The indefinite integral is as follows:

$$\int \sqrt{1 - z^2}\,dz = \tfrac{1}{2}z\sqrt{1 - z^2} + \tfrac{1}{2}\tan^{-1}\frac{z}{\sqrt{1 - z^2}}\,. \tag{7}$$

It might help to draw a right triangle with the opposite side of the angle as z and the adjacent side as $\sqrt{1 - z^2}$. Then, as $z \to 1$, the angle goes to $\pi/2$.