Math Diversion Problem 780

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He who would ride two camels, finds he can ride neither.
— From an old movie

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Source: https://www.youtube.com/watch?v=NGYX6gtObec
Title:  Introduction to conformal field theory, Lecture 1
          Part 2
 Presenter: Tobias Osborne
 (Read-along notes and a problem to solve.)
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1 Problem

Given the relation (in a conformal field theory)

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}\,, \tag{1}$$

show that

$$\big(\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu\big)(\partial \cdot \epsilon) = 0\,, \tag{2}$$

where

$$\Box \equiv \partial^\rho \partial_\rho\,. \tag{3}$$

2 Set up for conformal field theory in d dimensions

First, we define a manifold to work on. Let our manifold M be $\mathbb{R}^{(p,q)}$, where p is the number of basis vectors that square positive, and q is the number of basis vectors that square negative. [$p + q = d$]

Now for the metric:

$$\eta_{\mu\nu} = g_{\mu\nu}\,. \tag{4}$$

Note

$$g^{\mu\nu} g_{\mu\nu} = \eta^\mu_\mu = \delta^\mu_\mu = d\,. \tag{5}$$

Note: See Problem 775 for additional set-up information.

3 Solution

Note: We assume the commutivity of partial derivatives!

Let's rewrite (1) to

$$\partial_\mu \epsilon_\rho + \partial_\rho \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\rho} \,. \tag{6}$$

Next, we multiply through both sides by ∂^ρ and sum:

$$\partial_\mu \partial^\rho \epsilon_\rho + \partial^\rho \partial_\rho \epsilon_\mu = \frac{2}{d}\eta_{\mu\rho}\partial^\rho(\partial \cdot \epsilon) \,, \tag{7}$$

which simplifies to

$$\partial_\mu(\partial \cdot \epsilon) + \Box \epsilon_\mu = \frac{2}{d}\partial_\mu(\partial \cdot \epsilon) \,, \tag{8}$$

which can be rearranged to

$$\Big(1 - \frac{2}{d}\Big)\partial_\mu(\partial \cdot \epsilon) + \Box \epsilon_\mu = 0 \,. \tag{9}$$

Now, we multiply through both sides by ∂_ν:

$$\Big(1 - \frac{2}{d}\Big)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box \partial_\nu \epsilon_\mu = 0 \,. \tag{10}$$

That looks hopeful. Now, let's write this last equation with μ and ν interchanged:

$$\Big(1 - \frac{2}{d}\Big)\partial_\nu \partial_\mu(\partial \cdot \epsilon) + \Box \partial_\mu \epsilon_\nu = 0 \,. \tag{11}$$

On adding these last two equations together, we have that

$$2\Big(1 - \frac{2}{d}\Big)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box(\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu) = 0 \,. \tag{12}$$

Then, drawing on (1) again, we get

$$2\Big(1 - \frac{2}{d}\Big)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box\big(\frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}\big) = 0 \,. \tag{13}$$

Multiplying through by $d/2$ gives us

$$(d-2)\partial_\mu \partial_\nu(\partial \cdot \epsilon) + \Box(\partial \cdot \epsilon)\eta_{\mu\nu} = 0 \,, \tag{14}$$

which simplifies down to

$$\big(\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu\big)(\partial \cdot \epsilon) = 0 \,. \tag{15}$$

So, what was hard for me? 1) It took me a while to realize that I should introduce the third index ρ to get to (7). 2) It also took me a while to realize that I should double dip into (1), so that's why I interchanged μ and ν to get (11).