Math Diversion Problem 784

P. Reany

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Average talent, plus hard work and dedication,
will always beat talent by itself.
— Clinton Anderson

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Source: The Ether of Great Mathematical Ideas
Title:  Ellipses, Directrix (Part 3)
 Presenter: Patrick
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1 Introduction

Our first construction of the ellipse was in Article One, in which we used the method of drawing a curve on a plane with a pencil constrained to a string of fixed length and attached at two points in the plane (see Fig. 1.). Our first task this time will be to construct an ellipse by the 'directrix' method of construction and then show that this construction is equivalent to that of the previous one, which gave us the standard equation in the x, y-plane

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \,. \tag{1}$$

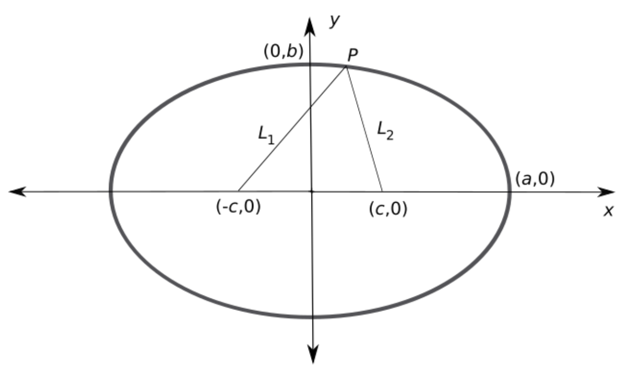

Figure 1. The ends of a string are attached to a flat surface at points $(-c, 0)$ and $(c, 0)$ shown as lying on the x-axis. The points at which the curve crosses

the x-axis are at $(a, 0)$ and $(-a, 0)$. It crosses the y-axis at $(0, b)$ and $(0, -b)$ Of course, the origin of the coordinate system lies halfway between the fixed points, $(-c, 0)$ and $(c, 0)$, which are known as the *foci* of the ellipse.

If we assume that the string is inextensible during this operation, then the total length of the string is constant (say, equal to L) and this is required for our analysis.

We're defining the curve that can be constructed by the above procedure as an *ellipse*.

In the first paper we proved that

$$c^2 = a^2 - b^2 \,. \tag{2}$$

We also defined the eccentricity number as

$$e \equiv \frac{c}{a} \,. \tag{3}$$

On eliminating c between these last two equations, we get

$$b^2 = a^2(1 - e^2) \,. \tag{4}$$

2 Much Ado About Horizontally Shifting the Center Point of the Ellipse

Don't be afraid of the greek letter ξ as it's just being used as a real number here. To horizontally shift the center of the ellipse by ξ to the right, one uses the equation

$$\frac{(x - \xi)^2}{a^2} + \frac{y^2}{b^2} = 1 \,. \tag{5}$$

(The standard symbol to use for horizontal shifting is h and k for vertical shifting.)

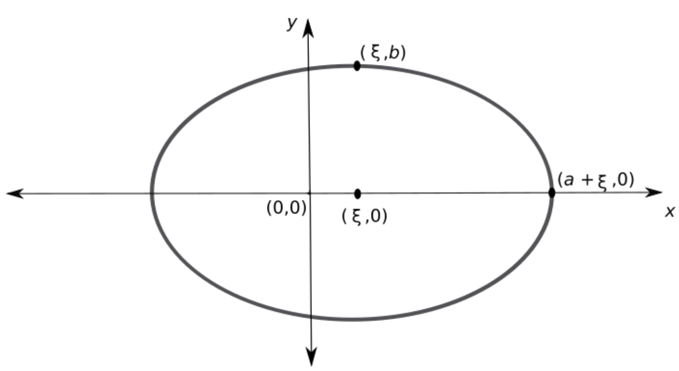

Figure 2. In this ellipse, the center is placed at the point $(\xi, 0)$, rather than at the origin, and ξ is any real number.

3 Construction of an Ellipse by Use of a Directrix

Our first task is to pick some real number e satissfying the constraint

$$0 < e < 1\,, \tag{6}$$

where e will play the role of the eccentricity of the ellipse.

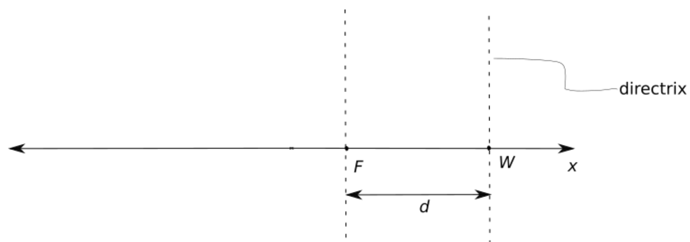

Figure 3. The point F will become the focus of the ellipse nearest to the directrix.

1) We place an origin of coordinates at point F. 2) We wish to identify all points P in the x, y-plane that satisfy the equation

$$e = \frac{|\,PF\,|}{|\,PM\,|}\,. \tag{7}$$

With origin at the focus F, the point P will have rectangular coordinates (x, y) or polar coordinates (r, θ).

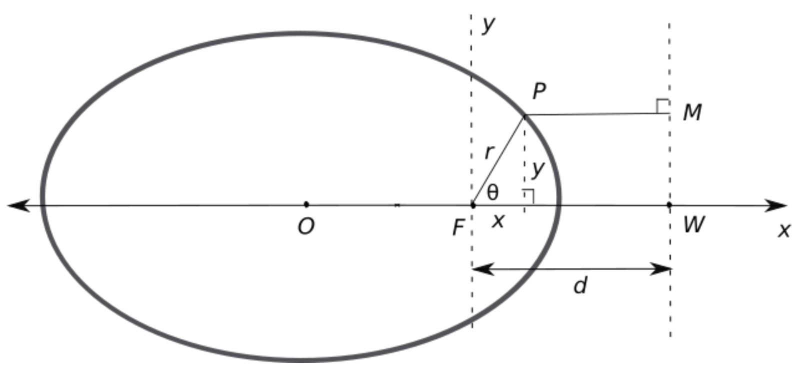

Figure 4. In this ellipse, the center is placed at the point O. We place an origin of rectangular coordinates at point F.

Now, we assign of a polar coordinate system to be the point F in the plane. Thus, by our construction

$$\begin{aligned} d &= r\cos\theta + |\,PM\,| \\ &= r\cos\theta + e^{-1}\,|\,PF\,| \quad \text{from (7)} \\ &= r\cos\theta + e^{-1} r\,. \end{aligned}$$

Hence,

$$ed = r(e\cos\theta + 1)\,, \tag{8}$$

or

$$r = \frac{ed}{1 + e\cos\theta}\,. \tag{9}$$

Our task now is to prove that the object we just constructed is an ellipse by comparing it to the previous construction. We are this time going to place the origin of coordinates at the focus point F and determine the coordinates of the center of the ellipse in this new coordinate system. And also show that our Equation (8) will convert into an equation of the form (5).

Let's recall that

$$x = r\cos\theta\,, \tag{10a}$$

$$y = r\sin\theta\,, \tag{10b}$$

$$r^2 = x^2 + y^2\,. \tag{10c}$$

So, from (8), we get

$$ed = ex + r\,, \tag{11}$$

or

$$r = e(d - x)\,. \tag{12}$$

Squaring this we get

$$r^2 = e^2(d^2 - 2dx + x^2)\,. \tag{13}$$

Using (10c), we get

$$x^2 + y^2 = e^2(d^2 - 2dx + x^2)\,. \tag{14}$$

Skipping a few steps, we get

$$\left(x + \frac{e^2 d}{1 - e^2}\right)^2 \left(\frac{(1 - e^2)^2}{e^2 d^2}\right) + \frac{y^2}{1 - e^2}\left(\frac{(1 - e^2)^2}{e^2 d^2}\right) = 1\,. \tag{15}$$

One more step and we're there:

$$\frac{\left(x - \left[-\dfrac{e^2 d}{1 - e^2}\right]\right)^2}{\left(\dfrac{ed}{(1 - e^2)}\right)^2} + \frac{y^2}{\left(\dfrac{ed}{\sqrt{1 - e^2}}\right)^2} = 1\,. \tag{16}$$

By comparing parts, we get that

$$a = ed/(1 - e^2)\,, \tag{17a}$$

$$b = ed/\sqrt{1 - e^2}\,, \tag{17b}$$

$$\xi = -e^2 d/(1 - e^2)\,. \tag{17c}$$

We can 'complete' this list of relations by solving (17a) for d:

$$d = \frac{a(1 - e^2)}{e} \, . \tag{18}$$

We have a consistency check we can place on this calculation for d: Returning to Fig. 4, we see that

$$2a = r_0 + r_\pi \, , \tag{19}$$

where $r_0 = r(\theta = 0)$ and $r_\pi = r(\theta = \pi)$ and $r(\theta)$ is determined from Eq. (9):

$$r_0 = ed/(1 + e) \, , \tag{20a}$$

$$r_\pi = ed/(1 - e) \, . \tag{20b}$$

Substituting these last two values into (19) and solving for d, we get

$$d = \frac{a(1 - e^2)}{e} \, . \tag{21}$$

4 What are the coordinates of W?

For a coordinate system whose origin is at the center of the ellipse, the coordinates of the point W is $(a^2/c, 0)$. So, I'll prove this now.

We know that $a = ed/(1 - e^2)$ and $c = e^2 d/(1 - e^2)$. From this we get that

$$\frac{a^2}{c} = \frac{d}{1 - e^2} \, . \tag{22}$$

If we represent the x coordinate of W as $\overline{x}$, then

$$\overline{x} = c + d = \frac{e^2 d}{1 - e^2} + d = \frac{d}{1 - e^2} = \frac{a^2}{c} \, . \tag{23}$$

5 Conclusion

The way to interpret Eq. (16) is to think of the ellipse being moved to the left a distance of

$$c = |\, \xi \,| = e^2 d/(1 - e^2) \, . \tag{24}$$

Or, alternatively, we can think of the effect of ξ being negative as moving the coordinate system to the right.