Math Diversion Problem 786

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Anything you can solve is interesting.
— Tobias Osborne

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Source: https://www.youtube.com/watch?v=NGYX6gtObec
Title:  Introduction to conformal field theory, Lecture 1
          Part 3
 Presenter: Tobias Osborne
 (Read-along notes and a problem to solve.)
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1 Problem

Given the relations (in a conformal field theory)

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}\,, \tag{1}$$

and

$$\big(\eta_{\mu\nu}\Box + (d-2)\partial_\mu \partial_\nu\big)(\partial \cdot \epsilon) = 0\,, \tag{2}$$

where

$$\Box \equiv \partial^\rho \partial_\rho\,, \tag{3}$$

show that, if we assume that ϵ^μ has a solution in the form

$$\epsilon^\mu = \omega^\mu{}_\nu x^\nu\,, \tag{4}$$

then $\omega^\mu{}_\nu$ is an antisymmetric tensor for condition $d > 2$.

2 Set up for conformal field theory in d dimensions

First, we define a manifold to work on. Let our manifold M be $\mathbb{R}^{(p,q)}$, where p is the number of basis vectors that square positive, and q is the number of basis vectors that square negative. [$p + q = d$]

Now for the metric:

$$\eta_{\mu\nu} = g_{\mu\nu}\,. \tag{5}$$

Note

$$g^{\mu\nu} g_{\mu\nu} = \eta^\mu_\mu = \delta^\mu_\mu = d\,. \tag{6}$$

Note: See Problem 786 for additional set-up information.

3 Solution

Note: We assume the commutivity of partial derivatives!

Let's put our ansatz solution (4) into (1) to get

$$\partial_\mu(\omega_{\nu_\beta} x^\beta) + \partial_\nu(\omega_{\mu_\beta} x^\beta) = \frac{2}{d}(\partial_\beta \omega^\beta{}_\xi x^\xi)\eta_{\mu\nu}\,. \tag{7}$$

From this we get

$$\omega_{\nu_\beta} \delta^\beta_\mu + \omega_{\mu_\beta} \delta^\beta_\nu = \frac{2}{d}\omega^\beta{}_\xi \delta^\xi_\beta \eta_{\mu\nu}\,. \tag{8}$$

which simplifies further to

$$\omega_{\mu\nu} + \omega_{\nu\mu} = \frac{2}{d}\,\omega^\beta{}_\beta \eta_{\mu\nu}\,. \tag{9}$$

Let

$$C \equiv \omega^\beta{}_\beta\,, \tag{10}$$

then (9) becomes

$$\omega_{\mu\nu} + \omega_{\nu\mu} = \frac{2}{d}\,C\,\eta_{\mu\nu}\,. \tag{11}$$

If $C \neq 0$ then

$$\omega_{\mu\nu} + \omega_{\nu\mu} \propto \eta_{\mu\nu}\,, \tag{12}$$

else

$$\omega_{\mu\nu} + \omega_{\nu\mu} = 0\,, \tag{13}$$

in which case

$$\omega_{\mu\nu} = -\omega_{\nu\mu}\,, \tag{14}$$

and $\omega_{\mu\nu}$ is an antisymmetric tensor.