Math Diversion Problem 787

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September 5, 2025

The four character traits that define a person are:
1) Goals & Commitments
2) Vices & Spiritual Victories
3) Experiences & Wisdom
4) Personality & Integrity

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Source: https://www.youtube.com/watch?v=z3qSpr1UrHs
Title:  Complex Numbers Gone Wild | P 568
 Presenter: aplusbi
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1 Problem

Given the relation

$$z^z = -1\,, \tag{1}$$

solve for the values of z.

2 Solution

This looks like a job for the Lambert W function![1]

Our first task is to 'complexify' -1.

$$z^z = -1 = e^{\pi i} = e^{\pi i} e^{2\pi i m} = e^{\pi i + 2\pi i m} \quad \text{where} \quad m \in \mathbb{Z}\,, \tag{2}$$

and that must be the wild part. Next, we take the natural logarithm across this last equation, to get

$$z \ln z = \pi i + 2\pi i m \quad \text{where} \quad m \in \mathbb{Z}\,. \tag{3}$$

Then we apply the Lambert W function to this to get

$$\ln z = W_n(\pi i + 2\pi i m) \quad \text{where} \quad m \in \mathbb{Z}\,, \tag{4}$$

[1]See the Appendix for a short write up on the Lambert W function.

where n is certain integers to be revealed.

On raising e to this last equation, we have that

$$z = e^{W_n(\pi i(1+2m))} \quad \text{where} \quad m \in \mathbb{Z}\,. \tag{5}$$

According to WolframAlpha, n will take on only a few values, such as $-1, 0, 1$. It also claims that one of these will give us

$$z = -1\,, \tag{6}$$

as the real solution. But it's also the trivial solution because we could have figured that out by inspection of (1).

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{7}$$

then

$$z = W(B)\,, \tag{8}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{9}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{10}$$

The following is the 'Lambert W function base s'[2], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{11}$$

which looks very similar to (7). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{12}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{13}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma) e^{W_n(\gamma)}\,. \tag{14}$$

[2]This notation I invented myself.