

Math Diversion Problem 791

P. Reany

September 8, 2025

A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell, [TV
show *Harry O*]

Source: <https://www.youtube.com/watch?v=0aoeB6xkqhE>
Title: Unexpected Symmetry
Presenter: Math Out Loud

1 Problem

Show that the function

$$f(x) = \ln(x + \sqrt{x^2 + 1}) \quad (1)$$

is an odd function, and then find $f^{-1}(x)$.

2 Preparation

A hyperbolic identity:

$$\cosh^2 u = \sinh^2 u + 1 \quad (2)$$

Note: $\cosh u$ is an even function, and $\sinh u$ is an odd function.

3 Solution

Let

$$x = \sinh u. \quad (3)$$

Then (1) becomes

$$f(x(u)) = \ln(\sinh u + \cosh u) = \ln e^u = u. \quad (4)$$

Therefore,

$$f(x) = \sinh^{-1} x. \quad (5)$$

Thus, $f(x)$ is an odd function. Why? Because when we functionally compose a function with its inverse, we need it to act as an identity on its argument. But this will only happen if the function and its inverse are both odd.

Anyway

$$f^{-1}(x) = \sinh x . \tag{6}$$