Math Diversion Problem 792

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> Risus est fructus doctrinae.
> (Laughter is the fruit of learning.)
> — Copilot

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Source: https://www.youtube.com/watch?v=Dbs6qhSVU5w
Title:  The Journey to the Infinite Power Tower
 Presenter: blackpenredpen
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1 Problem

Given the relation

$$x^{x^3} = 2 \tag{1}$$

solve for the values of x.

2 Solution

Out of habit, I'll use the method of alpha substitution. Let

$$x = 2^\alpha \,. \tag{2}$$

Then (1) becomes

$$(2^\alpha)^{2^{3\alpha}} = 2^1 \,. \tag{3}$$

After equating exponents, we get

$$\alpha 2^{3\alpha} = 1 \,. \tag{4}$$

Now, it looks like a job for the Lambert W function.[1] So, multiply through by 3:

$$3\alpha 2^{3\alpha} = 3 \tag{5}$$

and take the Lambert W function across it, to get

$$3\alpha = W_{(2)}(3) = \frac{W(3 \cdot \ln 2)}{\ln 2} \tag{6}$$

[1]See the Appendix.

Thus,

$$\alpha = W_{(2)}(3) = \frac{W_n(3 \cdot \ln 2)}{3 \ln 2} \tag{7}$$

To get x, we go back to (2):

$$x = 2^{\frac{W_n(3 \cdot \ln 2)}{3 \ln 2}} \,. \tag{8}$$

Now we tweak the problem a bit to

$$x^{x^3} = 3 \tag{9}$$

and then solve for the values of x, but this time, all we have to do is to replace the '2' in (8) by '3', to get

$$x = 3^{\frac{W_n(3 \cdot \ln 3)}{3 \ln 3}} \,. \tag{10}$$

Now, for $n = 0$, we have the real solution

$$x = 3^{\frac{\ln 3}{3 \ln 3}} = 3^{1/3} \,. \tag{11}$$

Let's redo (1) a different way. First, take the natural log across it:

$$x^3 \ln x = \ln 2 \,. \tag{12}$$

Then multiply through by 3:

$$x^3 \ln x^3 = 3 \ln 2 \,. \tag{13}$$

Take the Lambert W_0 function across this, to get

$$\ln x^3 = W_0(3 \ln 2) \,. \tag{14}$$

Solving for x, we have that

$$x = e^{\frac{1}{3} W_0(3 \ln 2)} \,. \tag{15}$$

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$z e^z = B \,, \tag{16}$$

then

$$z = W(B) \,, \tag{17}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{18}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{19}$$

The following is the 'Lambert W function base s'[2], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{20}$$

which looks very similar to (16). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{21}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{22}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma) e^{W_n(\gamma)}\,. \tag{23}$$

[2]This notation I invented myself.