Math Diversion Problem 793

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Success is failing 19 times and succeeding the 20th.
— Julia Andrews

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Source: https://www.youtube.com/watch?v=kL8srfaTw8w
Title:  Absolute value equation
 Presenter: Math Out Loud
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1 Problem

Given the relation

$$x\,|\,x\,|+1=3\,|\,x\,|\;, \tag{1}$$

how many solutions for x are there?

I haven't done a problem like this in a while.

2 Solution

Just algebra. My plan is to find all solutions for the subdomain $x \geq 0$ (Region I) and then for the subdomain $x < 0$ (Region II), and then count the number of distinct solutions combined.

In Region I, where $x \geq 0$, we can replace (1) by

$$x^2+1=3x\,, \tag{2}$$

or

$$x^2-3x+1=0\,, \tag{3}$$

which has roots $x=(3\pm\sqrt{5})/2$. So that's two roots.

In Region II, where $x < 0$, we can replace (1) by

$$-x^2+1=-3x\,, \tag{4}$$

or

$$x^2 - 3x - 1 = 0\,, \tag{5}$$

which has roots $x = (3 \pm \sqrt{13})/2$. So that's two more roots.

So that's a total of four possible solution to (1). We need to test them against (1). Both roots from Region I work because they are both greater than zero.

But let's make it a bit easier. Rewrite (1) as

$$(x - 3)\,|\,x\,| = -1\,. \tag{6}$$

Hence, the value of any root must be less than three. In Region I, they must also be greater than or equal to zero. Both roots in Region I satisfy these two constraints.

What about Region II? The root here must be both less than zero and less than three, which is just less than zero. The root $x = (3 - \sqrt{13})/2$ is less than zero, but the root $x = (3 + \sqrt{13})/2$ is greater than zero, so we eliminate it.

So, that leaves just three solutions

$$x \in \left\{(3 - \sqrt{5})/2, (3 + \sqrt{5})/2,\; (3 - \sqrt{13})/2\right\}. \tag{7}$$

3 Postscript

How did I get Eq. (5) from Eq. (1)? This is the region in which $x < 0$, and we want to drop the absolute value signs, leaving the equation the "same." But how should we think about this? We're going to go term-by-term. So, in (1) the first term on the LHS is $x\,|\,x\,|$. In Region II this has a negative value. Our job is to replace this term by something without absolute value signs, yet has the same magnitude and sign. Okay, using just x^2 will give us the correct magnitude but the wrong sign, so the fix is to put a minus sign in front of it to get $-x^2$.

We don't have to do anything to the second term on the LHS because it's a constant.

Now for the term on the RHS. Its value is always nonnegative. So the term $3x$ has the correct magnitude, but the wrong sign, so once again the fix is to put a negative sign in front of it. The result is Eq. (4), which is the same as (5).