Math Diversion Problem 794

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September 12, 2025

The human mind has never invented a labor-saving
machine equal to algebra.
— J. Willard Gibbs

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Source: The Ether of Great Mathematical Ideas
Title:  Reflection off of a surface
 Presenter: Patrick
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1 Problem

Our job is to use geometric algebra find a formula for the reflection of an incoming light ray off of a surface. The figure for it is below.

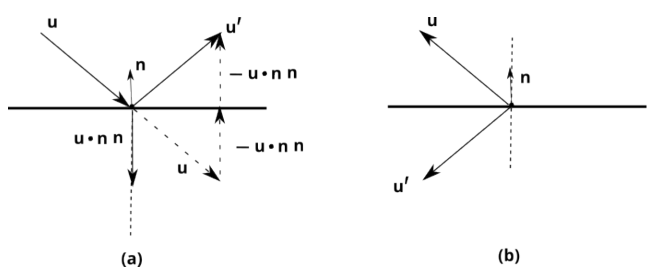

Figure 1. In the figure on the left, the vector $\mathbf{u}$, of arbitrary length, reflects off the horizontal surface (think ricochet), which has unit normal $\mathbf{n}$. The reflected vector is $\mathbf{u}'$. If, instead, you are interested in the reflection of $\mathbf{u}$ *through* the horizontal surface (see figure on the right), then $\mathbf{u} \to -\mathbf{u}$ and thus $\mathbf{u}' \to -\mathbf{u}'$.

Bear with me to see the method to my madness. If we take a copy of $\mathbf{u}$ and translate it from the second quadrant to the fourth quadrant, we notice two things. This vector has the correct horizontal length, but its projection onto the vertical axis, given algebraically by $\mathbf{u} \cdot \mathbf{n}\,\mathbf{n}$, in the wrong direction.

But why did I claim that the projection onto the vertical axis is given by $\mathbf{u} \cdot \mathbf{n}\,\mathbf{n}$ rather than by $-\mathbf{u} \cdot \mathbf{n}\,\mathbf{n}$? Sure, the $\mathbf{n}$ vector is up but this vector's projection is down because $\mathbf{u}\cdot\mathbf{n} < 0$. And this is the point: to get from $\mathbf{u}$ to $\mathbf{u}'$, we just need to use simple vector addition to the vectors in the figure. Thus,

$$\mathbf{u}' = \mathbf{u} - 2\mathbf{u}\cdot\mathbf{n}\,\mathbf{n}\,. \tag{1}$$

2 So where's the geometric algebra?

You noticed that, huh? First, we need an identity. For $\mathbf{a}$ and $\mathbf{b}$ vectors

$$\mathbf{b}\mathbf{a} = 2\mathbf{b}\cdot\mathbf{a} - \mathbf{a}\mathbf{b}\,. \tag{2}$$

I guess the simplest thing to do now is to just state the formula in geometric algebra and then prove it. So here it is:

$$\mathbf{u}' = -\mathbf{n}\mathbf{u}\mathbf{n}\,. \tag{3}$$

Proof:

$$\begin{aligned}
\mathbf{u}' &= -\mathbf{n}\mathbf{u}\mathbf{n} \\
&= -\mathbf{n}[\,2\mathbf{u}\cdot\mathbf{n} - \mathbf{n}\mathbf{u}\,] \\
&= -2\mathbf{u}\cdot\mathbf{n}\mathbf{n} + \mathbf{n}^2\mathbf{u} \\
&= -2\mathbf{u}\cdot\mathbf{n}\mathbf{n} + \mathbf{u} \\
&= \mathbf{u} - 2\mathbf{u}\cdot\mathbf{n}\mathbf{n}\,,
\end{aligned} \tag{4}$$

where we used the fact that $\mathbf{n}$ is a unit vector, so $\mathbf{n}^2 = 1$.

If $\mathbf{n}$ is not a unit vector, we have a similar formula for $\mathbf{u}'$, either

$$\mathbf{u}' = -\mathbf{n}^{-1}\mathbf{u}\mathbf{n} \quad \text{or} \quad \mathbf{u}' = -\mathbf{n}\mathbf{u}\mathbf{n}^{-1}\,. \tag{5}$$

3 What's the point?

Surely, there will be cases where it's easier to use (1) than to use (3). And, of course, this is correct, but now we have a choice. Perhaps even more to the point is the fact that the composition of reflections is a rotation.

Let $\mathbf{n}$ and $\mathbf{m}$ be arbitrary unit vectors in 3-space. Now, we reflect $\mathbf{u}$ first by $\mathbf{n}$ and then by $\mathbf{m}$:

$$\begin{aligned}
\mathbf{u}'' &= -\mathbf{m}\mathbf{u}'\mathbf{m} \\
&= -\mathbf{m}(-\mathbf{n}\mathbf{u}\mathbf{n})\mathbf{m} \\
&= (\mathbf{m}\mathbf{n})\mathbf{u}(\mathbf{n}\mathbf{m}) \\
&= (\mathbf{n}\mathbf{m})^\dagger\mathbf{u}(\mathbf{n}\mathbf{m}) \\
&= R^\dagger\mathbf{u}R
\end{aligned} \tag{6}$$

where

$$R = \mathbf{n}\mathbf{m} \tag{7}$$

is referred to as either a quaternion or a spinor of the geometric algebra.