Math Diversion Problem 795

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> The human mind has never invented a labor-saving machine equal to algebra.
> — J. Willard Gibbs

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Source: https://www.youtube.com/watch?v=G-BQFUsFIZM
Title:  A Nice Exponential Equation
 Presenter: SyberMath
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1 Problem

Given the relation

$$x^{x^{x+1}} = 265\,, \tag{1}$$

solve for the real values of x.

2 Solution

First off, let's take note that $256 = 2^8$. This probably means that we're in the territory of special cases. If that's so, one might assume that x is a power of 2 and proceed from there. But you can't get good at math by only solving special cases.

Let's take the natural logarithm across (1), to get

$$x^{x+1} \ln x = 8 \ln 2\,. \tag{2}$$

Now let's take the Lambert W function across this

$$W(x^{x+1} \ln x) = W(8 \ln 2) \; = W(4 \ln 4)\,. \tag{3}$$

Now, what would ever process me to end up with this mess? Simple. I happen to know that there are formulas for both the LHS and the RHS (see the appendix on the Lambert W function).

So, by applying those formulas, we get

$$x \ln x = \ln 4 = \ln 2^2 = 2 \ln 2\,. \tag{4}$$

If we apply Lambert again, we have that

$$\ln x = \ln 2\,. \tag{5}$$

And this leaves us with

$$x = 2\,. \tag{6}$$

This answer isn't surprising, as we expected to get a power of 2.

I just want to point out that these two Lambert formulas will not work for $W_n(y^{y+1}\ln y)$ and for $W_n(y\ln y)$ for arbitrary values of n, but they **do** work for $n = 0$.

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{7}$$

then

$$z = W(B)\,, \tag{8}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

This will be handy for this one!

$W_0(x^{x+1}\ln x) = x\ln x$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y\ln y = B\,, \tag{9}$$

then

$$\ln y = W(y\ln y) = W(B)\,. \tag{10}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{11}$$

which looks very similar to (7). Then

$$x = W_s(xs^x) \equiv \frac{W(A\ln s)}{\ln s}\,. \tag{12}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A\ln e)}{\ln e} = W(A)\,, \tag{13}$$

[1]This notation I invented myself.

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{14}$$