# Math Diversion Problem 797

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Chop your own wood and it will warm you twice.
— Henry Ford

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Source: The Ether of Great Mathematical Ideas
Title:  Reflection off of a curved surface
 Presenter: Patrick
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## 1 Problem

This paper uses Geometric Algebra to solve for the curve that focuses parallel rays to a single point. The curve will be shown to be a parabola and the point its focus. A basic knowledge of geometric algebra and first-year calculus is assumed. The previous article on geometric algebra Math Diversion Problem 794 might be of assistance.

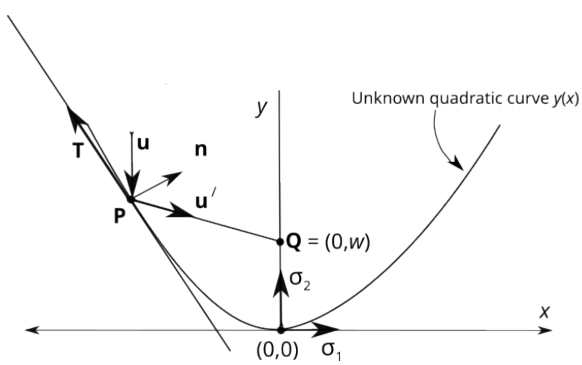


Figure 1. Our so-called unknown curve is a parabola, but we have to prove that and solve for its coefficients. The formula for $\mathbf{u}'$ is given by $-\mathbf{nun}$, where $\mathbf{u} = -\boldsymbol{\sigma}_2$ is a unit vector, and where $\mathbf{n}$ is the inward unit normal vector to the curve at the point $\mathbf{P}$. A priori, it's possible that there is no quadratic curve that suits the given requirements. Well, we'll see.

## 2 Introduction

Consider the limiting case of an infinitely shallow 2-D reflecting structure (curve) in $\mathbb{R}^2$. It will reflect all light rays that hit it by the rule *the angle of reflection is equal to the angle of incidence*. We demand that this reflecting structure sends all reflecting rays to the point $\mathbf{Q} = (0, w)$ with $w > 0$. We also demand that this reflecting structure pass through the point $(0, 0)$. Lastly, we'll assume that the equation of this curve is at most quadratic. The situation is depicted in the figure below.

## 3 Solution

For starters, we're assuming the curve to be a polynomial, so it's differentiable everywhere. Therefore, it has a well-defined tangent line to every point $\mathbf{P}$ on the curve. By the way, since the point $\mathbf{P}$ is arbitrarily chosen, what we prove about the reflection at this point is true for every point on the curve. As for the curve, we're assuming it has the polynomial form:

$$f(x) = ax^2 + bx\,, \tag{1}$$

where the constant term is zero to satisfy the requirement that the curve goes through the origin. Anyway, the slope function $f'(x)$ is given by

$$f'(x) = 2ax + b\,. \tag{2}$$

The usual geometric algebra encoding for a reflection about a normal, unit vector $\mathbf{n}$ is given by

$$\mathbf{u}' = -\mathbf{nun}\,. \tag{3}$$

However, we will re-organize this by replacing $\mathbf{n}$ by $\mathbf{T}$, the unit tangent vector along the curve at $P$. So, let $\mathbf{I}$ be a unit bivector of the plane containing both $\mathbf{n}$ and $\mathbf{T}$. Now, we can rotate $\mathbf{n}$ into $\mathbf{T}$ by hitting $\mathbf{n}$ on the right by $\mathbf{I}$:

$$\mathbf{T} = \mathbf{nI}\,. \tag{4}$$

Next, we replace $\mathbf{n}$ by $\mathbf{T}$. Now, according to the fact that all vectors in a plane anticommute with bivectors of the plane and that $\mathbf{I}^2 = -1$, we can rewrite (3) in terms of $\mathbf{T}$:

$$\begin{aligned}
\mathbf{u}' &= -\mathbf{nun} \\
&= -\mathbf{n}(-\mathbf{I}^2)\mathbf{un} \\
&= (\mathbf{nI})(\mathbf{Iu})\mathbf{n} \\
&= -(\mathbf{nI})(\mathbf{uI})\mathbf{n} \\
&= -(\mathbf{nI})\mathbf{u}(\mathbf{In}) \\
&= (\mathbf{nI})\mathbf{u}(\mathbf{nI}) \\
&= \mathbf{TuT}\,.
\end{aligned} \tag{5}$$

Now, if you're concerned that I have been sloppy about the direction of $\mathbf{T}$ along the tangent line, one need not worry. Yes, it's true that when I calculate the tangent vector along the tangent line, provided by taking the derivative of $f(x)$, that the tangent vector might come out in one of two of two orientations along that line, which we can indicated by $\pm$ (and I'll probably be sloppy about that too, because there is a simple reason: $\mathbf{T}$ appears twice in the expression $\mathbf{TuT}$, so its sign won't matter).

And the conversion leaves us with

$$\mathbf{u}' = \hat{\mathbf{T}}(-\mathbf{u})\hat{\mathbf{T}}\,, \tag{6}$$

where I put in the hats to indicate that this $\mathbf{T}$ is a unit tangent vector, which technically should be there. (Although we'll soon see that we don't need them.)

**We're Ready to Go**

Now, to enforce that two vectors are parallel, we set their wedge product to zero. So, to force $\mathbf{u}'$ to lie along the difference vector $\pm(\mathbf{Q}-\mathbf{P})$, we must have that

$$\mathbf{u}' \wedge (\mathbf{P}-\mathbf{Q}) = 0\,, \tag{7}$$

which becomes

$$[\,\hat{\mathbf{T}}(\mathbf{u})\hat{\mathbf{T}}\,] \wedge (\mathbf{P}-\mathbf{Q}) = 0\,. \tag{8}$$

and clearly the sign of $(\mathbf{Q}-\mathbf{P})$ is irrelevant. Also, we can multiply through both sides by $|\,\mathbf{T}\,|^2$ to get

$$[\,\mathbf{T}(\mathbf{u})\mathbf{T}\,] \wedge (\mathbf{P}-\mathbf{Q}) = 0\,. \tag{9}$$

and it won't matter which sign we choose. Also, by construction, $\mathbf{u} = -\boldsymbol{\sigma}_2$.

$$[\,\mathbf{T}(\boldsymbol{\sigma}_2)\mathbf{T}\,] \wedge (\mathbf{P}-\mathbf{Q}) = 0\,. \tag{10}$$

Again, we ignored the sign on $\boldsymbol{\sigma}_2$ because the overall sign doesn't matter.

To finish up, we need to calculate $\mathbf{T}$ and $(\mathbf{P}-\mathbf{Q})$, and we'll do the latter first: By referring to the figure above, we get

$$\mathbf{P}-\mathbf{Q} = (x,y)-(0,w) = x\boldsymbol{\sigma}_1 + (y-w)\boldsymbol{\sigma}_2\,. \tag{11}$$

So, now we need a tangent vector $\mathbf{T}$.

$$\begin{aligned}\mathbf{T} &= \lim_{\Delta x\to 0}\frac{(\Delta x,\Delta y)}{\Delta x} = \lim_{\Delta x\to 0}\big(1,\frac{\Delta y}{\Delta x}\big)\\ &= \big(1,\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x}\big)\\ &= (1,f'(x))\,.\end{aligned}$$

On converting this to use explicit basis vectors, we have that

$$\mathbf{T} = (1,y') = \boldsymbol{\sigma}_1 + y'\boldsymbol{\sigma}_2\,. \tag{12}$$

Now we calculate $\mathbf{T}\,\boldsymbol{\sigma}_2\,\mathbf{T}$.

$$\begin{aligned}\mathbf{T}\,\boldsymbol{\sigma}_2\,\mathbf{T} &= (\boldsymbol{\sigma}_1 + y'\boldsymbol{\sigma}_2)\boldsymbol{\sigma}_2(\boldsymbol{\sigma}_1 + y'\boldsymbol{\sigma}_2)\\ &= (\boldsymbol{\sigma}_{12} + y')(\boldsymbol{\sigma}_1 + y'\boldsymbol{\sigma}_2)\\ &= -\boldsymbol{\sigma}_2 + 2y'\boldsymbol{\sigma}_1 + (y')^2\boldsymbol{\sigma}_2\\ &= 2y'\boldsymbol{\sigma}_1 + [\,(y')^2 - 1\,]\boldsymbol{\sigma}_2 \qquad (13)\end{aligned}$$

Having performed all the preparatory calculations, we'll get the values we need for $a$ and $b$ by substituting in the appropriate values in terms of $y, y', w$. So, let's proceed.

$$\begin{aligned}[\,\mathbf{T}\,\mathbf{u}\,\mathbf{T}\,] \wedge (\mathbf{P} - \mathbf{Q}) &= \big[\,2y'\boldsymbol{\sigma}_1 + [\,(y')^2 - 1\,]\boldsymbol{\sigma}_2\,\big] \wedge [\,x\boldsymbol{\sigma}_1 + (y - w)\boldsymbol{\sigma}_2\,]\\ &= 2y'(y-w)\boldsymbol{\sigma}_{12} - [\,(y')^2 - 1\,]x\,\boldsymbol{\sigma}_{12}\\ &= \{2y'(y-w) - [\,(y')^2 - 1\,]x\}\boldsymbol{\sigma}_{12}\\ &= 0\,. \qquad (14)\end{aligned}$$

So, if this procedure is to work, the constraints we need on $a$ and $b$ will be evident when we expand

$$2y'(y-w) - [\,(y')^2 - 1\,]x = 0\,, \qquad (15)$$

and then compare coefficients to (1). Substituting in from (1) and (2), we get, after much simplification,

$$2abx^2 + (-4aw + b^2 + 1)x - 2bw = 0\,. \qquad (16)$$

First, on comparing this to (1), we see that $-2bw = 0$, and since $w \neq 0$, then $b = 0$, leaving us with

$$(-4aw + 1)x = 0\,. \qquad (17)$$

On setting this last coefficient to zero, we get

$$w = \frac{1}{4a}\,, \qquad (18)$$

which is the expected result, although we need to solve for $a$ in this case, to get for our quadratic polynomial

$$f(x) = \frac{1}{4w}x^2\,, \qquad (19)$$

which, in standard form, would be written as

$$4w(y + k) = (x + h)^2\,, \qquad (20)$$

with $h = k = 0$, since the vertex is at point $(0, 0)$.