Math Diversion Problem 803

P. Reany

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Great math books need to start with intuition,
context, and simple examples, and from there
build up to formalism and proofs.
— Luca Di Beo

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Source: https://www.youtube.com/watch?v=i-Q74u_tPHo
Title:  Stanford math tournament algebra tiebreaker
 Presenter: blackpenredpen
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1 Problem

Given the relation

$$\frac{1}{\log_8 n} + \frac{1}{\log_n \frac{1}{4}} = -\frac{5}{2}\,, \tag{1}$$

solve for the integer values of n.

Hint: This problem shows a lot of powers of 2, so perhaps we should convert the logarithms to base 2.

Lemma 1:

$$\frac{1}{\log_a b} = \log_b a\,. \tag{2}$$

Lemma 2:

$$\log_a b = \frac{\log_c b}{\log_c a}\,. \tag{3}$$

Lemma 3:

$$\log_a b^{-1} = -\log_a b\,. \tag{4}$$

2 Solution

Eq. (1) becomes

$$\frac{1}{(\log_2 n)/(\log_2 8)} + \frac{1}{-\log_n 4} = -\frac{5}{2}\,, \tag{5}$$

or

$$\frac{\log_2 8}{\log_2 n} - \log_4 n = -\frac{5}{2}\,, \tag{6}$$

or

$$\frac{\log_2 2^3}{\log_2 n} - \frac{\log_2 n}{\log_2 4} = -\frac{5}{2}\,, \tag{7}$$

or

$$\frac{3}{\log_2 n} - \frac{\log_2 n}{2} = -\frac{5}{2}\,. \tag{8}$$

Multiply through by $-2\log_2 n$, to get

$$-6 + (\log_2 n)^2 = 5\log_2 n\,, \tag{9}$$

which can be put in standard quadratic form as

$$(\log_2 n)^2 - (5\log_2 n) - 6 = 0\,, \tag{10}$$

which can be factored to

$$(\log_2 n + 1)(\log_2 n - 6) = 0\,. \tag{11}$$

Therefore we have the two possible solutions:

$$\log_2 n = -1 \quad \text{or} \quad \log_2 n = 6\,. \tag{12}$$

Hence the possible solutions for n are:

$$n = -1/2 \quad \text{or} \quad n = 2^6 = 64\,, \tag{13}$$

of which only the latter is acceptable according to the given instructions.