Math Diversion Problem 810

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September 22, 2025

You cannot read mathematics the way you read a
novel. If you zip through a page in less than an
hour, you are probably going too fast.
— Sheldon Axler
(from *Linear Algebra Done Right*)

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Source:  The Ether of Great Mathematical Ideas
Title:  The Commutative Diagram
 Presenter: Patrick
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1 Setup

The Commutative Diagram is a subclass of path-independent diagram. In the figure below, think of the vertices as objects of some sort, and the arrows between them as representing morphisms between them. So, x, x', y, y' are the objects and f, g, η_x, η_y are morphisms between the objects.

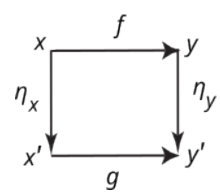

Figure 1. The commutative diagram in its simplest form: We start at x and end up at y'. Note that if the objects are sets, the arrows are just functions, by convention.

I called the figure in Fig. 1 a "commutative diagram," but that was premature. This is what will make the figure "commutative." If we start at x and take the path f followed by η_y is the same effect as starting at x and taking the path η_x followed by g, then the diagram is said to commute. In either case we end up at y'. Sometimes the figure is referred to as a "commutative square."

What we're doing is to take a composition of morphisms in two different ways, yet getting the same result. No doubt there ought to be a compositional equation for this, and the following is it.

$$\eta_y \circ f = g \circ \eta_x \,. \tag{1}$$

On either side of this equation, we start at the rightmost morphism and then move to the left, applying each successive morphism/s as we go. But if we are going to read it the normal way from left-to-right, we have to make an adjustment. In which case, it's typical to say for Eq. 1,

η_y after f is equal to g after η_x.

2 Problem

In the figure below, we find two commutative squares stuck together. In equational form this means that

$$\eta_y \circ f = g \circ \eta_x \,. \tag{2}$$

and that

$$\eta_{y'} \circ g = h \circ \eta_{x'} \,. \tag{3}$$

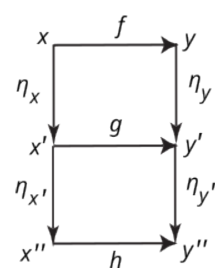

Figure 2. Don't let the added complications intimidate you.

So, the problem is to show that the rectangle with vertices x, y, x'', y'' commutes, with compositional equation

$$\eta_{y'} \circ \eta_y \circ f = h \circ \eta_{x'} \circ \eta_x \,, \tag{4}$$

where I have not included parentheses because we are assuming that the composition of morphisms is associative, but we may put in parentheses if we wish to.

3 Solution

Let's begin by appending morphism $\eta_{y'}$ on the left of Eq. (2):

$$\begin{aligned} \eta_{y'} \circ \eta_y \circ f &= \eta_{y'} \circ g \circ \eta_x \\ &= (\eta_{y'} \circ g) \circ \eta_x \\ &= (h \circ \eta_{x'}) \circ \eta_x \quad \text{from} \quad (3) \\ &= h \circ \eta_{x'} \circ \eta_x \,. \end{aligned} \tag{5}$$

Okay, that wasn't so hard, so what's the point of it? The point of it is to get some practice doing "arrow chasing" in category theory, if you should ever want or need to learn it.