Math Diversion Problem 811

P. Reany

September 23, 2025

Beware of the natural tendency to be self-sufficient.
In the good times it makes us arrogant;
and in the bad times, fearful.
— a proverb

```
Source:  https://www.youtube.com/watch?v=oN_iZB-K-Go
Title:  International Math Olympiads PROBLEM
 Presenter:  Learn with Christian Ekpo
```

1 Problem

Given the relation

$$t^5 = 9^t \,, \tag{1}$$

find the values for t.

2 Solution

This looks like a job for the Lambert W function.[1] Let start by taking the fifth root across (1):

$$t = \alpha^t \,. \tag{2}$$

where $\alpha \equiv 9^{1/5}$. Next, move α^t to the other side, to get

$$t\alpha^{-t} = 1 \,. \tag{3}$$

Then multiply through by -1, and we have that

$$-t\alpha^{-t} = -1 \,. \tag{4}$$

Now we apply the Lambert W_α function,

$$-t = W_\alpha(-1) = \frac{W_n(-\ln\alpha)}{\ln\alpha} = \frac{W_n(-\ln 9^{1/5})}{\ln 9^{1/5}} = 5\frac{W_n(-\frac{1}{5}\ln 9)}{\ln 9} \,. \tag{5}$$

Finally,

$$t_n = -5\frac{W_n(-\frac{1}{5}\ln 9)}{\ln 9} \,. \tag{6}$$

[1]For information about the Lambert W function, see the Appendix.

3 Appendix: Lambert

Sometimes I need to use the Lambert W function, which goes as follows: If

$$ze^z = B\,, \tag{7}$$

then

$$z = W(B)\,, \tag{8}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{9}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{10}$$

The following is the 'Lambert W function base s'[2], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{11}$$

which looks very similar to (7). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{12}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{13}$$

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma) e^{W_n(\gamma)}\,. \tag{14}$$

[2]This notation I invented myself.