Math Diversion Problem 813

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> The commitment to all truth is a test of character
> that God watches for in each of us.
> — a proverb

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Source:  https://www.youtube.com/watch?v=cJaxM9nvQrA
Title:  A logarithm equation - Viewer submitted problem
 Presenter:  Math Out Loud
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1 Problem

Given the relation

$$\log_x \alpha + \log_{ax} \alpha^2 + \log_{ax^2} \alpha^3 = 0\,, \tag{1}$$

where $a > 1$ and $\alpha > 1$, solve for x in terms of a and α. Note: I made both a and α different than originally posed.

2 A Lemma

$$\log_a y = \frac{\log y}{\log a}\,, \tag{2}$$

where I chose the default base of base ten, but one could use any positive number for the base as one wishes.

3 Solution

Using the lemma, (1) becomes

$$\frac{\log \alpha}{\log x} + \frac{\log \alpha^2}{\log ax} + \frac{\log \alpha^3}{\log ax^2} = 0\,, \tag{3}$$

which expands to

$$\frac{\log \alpha}{\log x} + \frac{2\log \alpha}{\log a + \log x} + \frac{3\log \alpha}{\log a + 2\log x} = 0\,. \tag{4}$$

Since $\alpha > 1$, $\log \alpha > 0$, hence we can divide through by $\log \alpha$ without worrying about the ramifications to the values of x.

$$\frac{1}{\log x} + \frac{2}{\log a + \log x} + \frac{3}{\log a + 2\log x} = 0\,. \tag{5}$$

So, α will play no role in the solution because the original equation is homogeneous.

Next, we do some simplifying. Let $y = \log x$ and $b = \log a$, then we have that

$$\frac{1}{y} + \frac{2}{b+y} + \frac{3}{b+2y} = 0\,. \tag{6}$$

Well, let's clear of fractions:

$$(b+y)(b+2y) + 2y(b+2y) + 3y(b+y) = 0\,. \tag{7}$$

According to WolframAlpha, this has solutions for y in terms of b as

$$y = \frac{1}{9}\big(-4 \pm \sqrt{7}\big)\,b\,, \tag{8}$$

which transforms to

$$\log x = \frac{1}{9}\big(-4 \pm \sqrt{7}\big)\,\log a\,, \tag{9}$$

And now we're ready to state x:

$$x_{\pm} = a\,10^{(-4\pm\sqrt{7})/9}\,. \tag{10}$$