

Math Diversion Problem 819

P. Reany

September 29, 2025

Every good mathematician is at least half a philosopher,
and every good philosopher is at least
half a mathematician.
— Gottlob Frege

Source: The Ether of Great Mathematical Ideas
Title: The tangent vector to a line in the plane
Presenter: Patrick

1 Problem

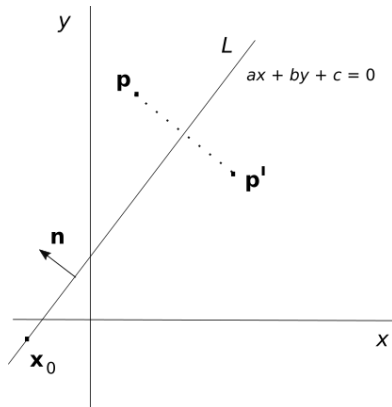
We are given the equation of a line L in the plane in standard form as¹

$$ax + by + c = 0. \quad (1)$$

Let \mathbf{p} be any point not on L . Our task is use geometric algebra to find the reflection on \mathbf{p} through the line L We'll call this point \mathbf{p}' . Refer to Fig. 1. Show that \mathbf{p}' can be found from the given information as

$$\mathbf{p}' = \mathbf{p} - 2(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{n} \mathbf{n}, \quad (2)$$

where \mathbf{x}_0 is any point on L .



¹We assume that $b \neq 0$. If it is, we launch an exception and set $\mathbf{n} = \pm \sigma_1$.

Figure 1. Point \mathbf{p} is a given point not on L . Vector \mathbf{n} is normal to L .

The point \mathbf{x}_0 is to be arbitrarily chosen on L .

The geometric-algebra equation for this problem is

$$\mathbf{p}' = \mathbf{x}_0 - \mathbf{n}(\mathbf{p} - \mathbf{x}_0)\mathbf{n}, \quad (3)$$

where \mathbf{x}_0 is arbitrary and can be determined from (8). The vector \mathbf{n} is a unit vector perpendicular to line L . It can also be determined from (8). The way to think about this equation is simple: To get to \mathbf{p}' , start at \mathbf{x}_0 and then add to it the reflection vector relative to line L .

2 Proof

Now, here's how we get \mathbf{n} :²

$$\mathbf{n} = \frac{a\boldsymbol{\sigma}_1 + b\boldsymbol{\sigma}_2}{\sqrt{a^2 + b^2}} = \frac{(a, b)}{\sqrt{a^2 + b^2}}. \quad (4)$$

The only other thing we need to begin the proof is this very basic identity:

$$\mathbf{a}\mathbf{b} = 2\mathbf{a} \cdot \mathbf{b} - \mathbf{b}\mathbf{a}. \quad (5)$$

Solution:

$$\begin{aligned} \mathbf{p}' &= \mathbf{x}_0 - \mathbf{n}(\mathbf{p} - \mathbf{x}_0)\mathbf{n} \\ &= \mathbf{x}_0 - \mathbf{n}[2(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{n} - \mathbf{n}(\mathbf{p} - \mathbf{x}_0)] \\ &= \mathbf{x}_0 - 2(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{n}\mathbf{n} + (\mathbf{p} - \mathbf{x}_0) \\ &= \mathbf{p} - 2(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{n}\mathbf{n}. \end{aligned} \quad (6)$$

■

3 Example problem

Let's do an example problem. Say our given line equation is

$$y = x + 1, \quad (7)$$

which in standard form becomes,

$$-x + y - 1 = 0. \quad (8)$$

Hence, $a = -1$ and $b = 1$. Thus,

$$\mathbf{n} = \frac{-\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2}{\sqrt{2}} = \frac{1}{\sqrt{2}}(-1, 1). \quad (9)$$

²See the Appendix for more detail.

If we are given the point $\mathbf{P} = (-1, 1)$ to be reflected through line L , then all we need to employ Eq. (6) is to first pick some point \mathbf{x}_0 on L . We chose for \mathbf{x}_0 the point $(-1, 0)$

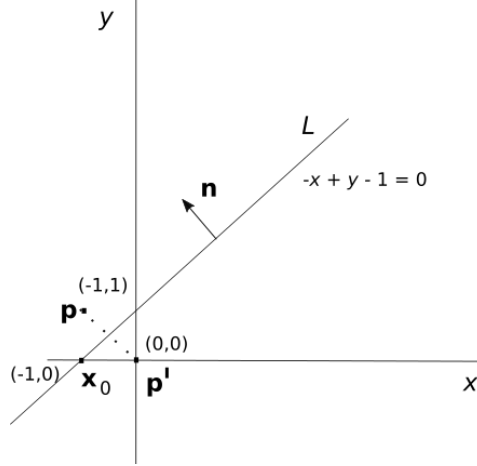


Figure 2. Point \mathbf{p} is a given point not on L . Vector \mathbf{n} is normal to L . The point \mathbf{x}_0 is to be arbitrarily chosen on L .

$$\begin{aligned}
 \mathbf{p}' &= \mathbf{p} - 2(\mathbf{p} - \mathbf{x}_0) \cdot \mathbf{n} \mathbf{n} \\
 &= (-1, 1) - 2[(-1, 1) - (-1, 0)] \cdot \frac{1}{\sqrt{2}}(-1, 1) \frac{1}{\sqrt{2}}(-1, 1) \\
 &= (-1, 1) - [(-1, 1) - (-1, 0)] \cdot (-1, 1) (-1, 1) \\
 &= (-1, 1) - [(0, 1) \cdot (-1, 1)](-1, 1) \\
 &= (-1, 1) - (-1, 1) \\
 &= (0, 0).
 \end{aligned} \tag{10}$$

4 Appendix

How to get \mathbf{n} ? Let \mathbf{t} be any tangent vector to the line

$$ax + by + c = 0. \tag{11}$$

This vector will be characterized by its slope — that is, its rise over its run, to use the old language; but otherwise its length doesn't matter.

The slope $\Delta y / \Delta x$ of the line in (11) is

$$\frac{\Delta y}{\Delta x} = \frac{-a}{b}. \tag{12}$$

Okay, so a vector that has the same rise-over-run as the line could be

$$\mathbf{t} = \Delta x \boldsymbol{\sigma}_1 + \Delta y \boldsymbol{\sigma}_2 = b \boldsymbol{\sigma}_1 - a \boldsymbol{\sigma}_2. \tag{13}$$

It's easy to verify that \mathbf{n} given by (4) satisfies the equation

$$\mathbf{n} \cdot \mathbf{t} = 0, \tag{14}$$

and thus the two vectors are orthogonal to each other.