Math Diversion Problem 820

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If you're not in love with the Truth, you could be
talked into believing almost anything.
— Author

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 Title:  Where a line intersects a plane
  Presenter:  Patrick
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1 Problem

This paper uses Geometric Algebra to solve for intersection point of a line and a plane in 3D. Knowledge of geometric algebra is assumed.

2 Introduction

We start with an arbitrary plane Π in 3D space and any line L that intersects the plane in exactly one point. From here on, how we proceed depends a lot on how the rest of the information we need is presented to us. For example, the line could be presented as a point and a normal to another plane, or as in the case I go over here, it could be represented by point and a direction vector. The plane could be represented by three distinct noncollinear points, or as $ax + by + cz + d = 0$, or, as in this problem, the plane is could be represented as a point on the plane and two distinct vectors 'in' the plane. In the latter case, one can either use the normal to the plane (obtained from the cross product of these two vectors) or use the wedge product of these two vectors.

Let $\mathbf{u}$ be the direction of the line in space. Let $\mathbf{P}$ be any point on L but not on Π and let $\mathbf{Q}$ be any point on Π but not on L. Let $\mathbf{S}$ be the point of intersection of L with Π. Lastly, let $\mathbf{v}_1$ and $\mathbf{v}_2$ be any two distinct vectors in the plane Π, which then 'span' the points in plane Π.

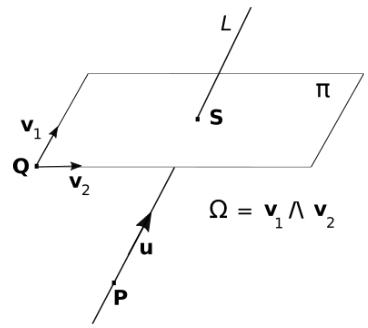

Figure 1. How hard is it to solve this problem using geometric algebra?

3 Solution

Now, we can't just claim that point $\mathbf{S}$ is on line L; we need an algebraic constraint to establish it. That constraint is

$$(\mathbf{S}-\mathbf{P}) \wedge \mathbf{u} = 0\,. \tag{1}$$

But $\mathbf{S}$ is also a member of plane Π, and to establish that, we need to find scalars α and β such that

$$\mathbf{S} = \mathbf{Q} + \alpha\mathbf{v}_1 + \beta\mathbf{v}_2\,. \tag{2}$$

This equation means that we start at the point $\mathbf{Q}$ and then go α in the $\mathbf{v}_1$ direction and then go β in the $\mathbf{v}_2$ direction. On combining these last two equations to eliminate $\mathbf{S}$ between them, we get

$$(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \mathbf{w}) \wedge \mathbf{u} = 0\,, \tag{3}$$

where we introduced the vector $\mathbf{w} \equiv \mathbf{Q} - \mathbf{P}$ for simplicity.

If now we wedge on the right of (3) by $\mathbf{v}_2$, we get[1]

$$(\alpha\mathbf{v}_1 + \mathbf{w}) \wedge \mathbf{u} \wedge \mathbf{v}_2 = 0\,. \tag{4}$$

But if we now wedge on the right of (3) by $\mathbf{v}_1$, we get

$$(\beta\mathbf{v}_2 + \mathbf{w}) \wedge \mathbf{u} \wedge \mathbf{v}_1 = 0\,. \tag{5}$$

We can solve (4) for α to get

$$\begin{aligned}\alpha &= -\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v}_2 / \mathbf{v}_1 \wedge \mathbf{u} \wedge \mathbf{v}_2 \\ &= -\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}_2) / \mathbf{v}_1 \cdot (\mathbf{u} \times \mathbf{v}_2)\,,\end{aligned} \tag{6}$$

[1]Trivectors in 3D space commute with all elements of $\mathcal{G}_3$, therefore division by a trivector makes sense.

which shows that α can be expressed as the ratio of two determinants.[2] Similarly, for β we get

$$\begin{aligned}\beta &= -\mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v}_1 / \mathbf{v}_2 \wedge \mathbf{u} \wedge \mathbf{v}_1 \\ &= -\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}_1) / \mathbf{v}_2 \cdot (\mathbf{u} \times \mathbf{v}_1)\,. \end{aligned} \tag{8}$$

Hence,

$$\mathbf{S} = \mathbf{Q} - \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}_2)}{\mathbf{v}_1 \cdot (\mathbf{u} \times \mathbf{v}_2)} \mathbf{v}_1 - \frac{\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}_1)}{\mathbf{v}_2 \cdot (\mathbf{u} \times \mathbf{v}_1)} \mathbf{v}_2\,. \tag{9}$$

[2]For example, to express $\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}_2)$ as a determinant,

$$\mathbf{w} \cdot (\mathbf{u} \times \mathbf{v}_2) = \det \begin{pmatrix} w_1 & w_2 & w_3 \\ u_1 & u_2 & u_3 \\ v_{21} & v_{22} & v_{23} \end{pmatrix}\,. \tag{7}$$

4 Conclusion

What about using a normal $\mathbf{N}$ to the plane? Okay, with it you can drop a perpendicular from $\mathbf{P}$ to the plane. Nevertheless, I'd probably just convert it to a bivector of the plane by

$$\Omega = i\mathbf{N}\,. \tag{10}$$

All I can say is that this technique of using a normal direction to define a plane does not generalize to dimensions of four or more, as there is no unique normal direction of a plane in 4D or in higher dimensions, yet a plane can always be represented by a bivector.