Math Diversion Problem 822

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> Mental toughness is essential to success.
> — Vince Lombardi

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Source: https://www.youtube.com/watch?v=_4S6bl_G2ak
 Title:  This Equation Breaks Your Brain! | P580
  Presenter:  aplusbi
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1 Problem

Given the relation

$$z \ln z = -\frac{\pi}{2}\,, \tag{1}$$

find the most general solution for z.[1]

This looks like a job for the Lambert W function! (See the Appendix.)

2 Solution

We begin by taking the Lambert W function across (1):

$$\ln z = W_n\big(-\frac{\pi}{2}\big) \quad \text{where } n \in \mathbb{Z}\,. \tag{2}$$

Next, we insert $e^{2\pi i m}$ into the mix:

$$\ln\left(z e^{2\pi i m}\right) = W_n\big(-\frac{\pi}{2}\big) \quad \text{where } m, n \in \mathbb{Z}\,, \tag{3}$$

which becomes

$$\ln z + 2\pi i m = W_n\big(-\frac{\pi}{2}\big) \quad \text{where } m, n \in \mathbb{Z}\,, \tag{4}$$

or

$$\ln z = W_n\big(-\frac{\pi}{2}\big) - 2\pi i m \quad \text{where } m, n \in \mathbb{Z}\,. \tag{5}$$

[1]I'm going to qualify this. One can get carried away replacing unity with $e^{2\pi i m}$ where m is an integer. So I'm going to place it where it might end up in the answer.

Now we exponentiate, to get

$$z = \exp\left\{W_n\left(-\frac{\pi}{2}\right)\right\} \quad \text{where}\, n \in \mathbb{Z}\,. \tag{6}$$

Seems like m just did not want to appear in the final answer. Anyway, for $n = 0$, we get

$$z_0 = \exp\{i\pi/2\} = i\,. \tag{7}$$

3 Appendix: Lambert W Function

Warning: The Lambert W function can be tricky because it's subtle. Please do not attempt to land any craft on Mars based on my calculations using the Lambert W function. We could both end up with Martian egg on our faces.

The Lambert W function goes as follows: If

$$ze^z = B\,, \tag{8}$$

then

$$z = W(B)\,, \tag{9}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.
Note:

$W_0(-\pi/2) = i\pi/2$.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B\,, \tag{10}$$

then

$$\ln y = W(y \ln y) = W(B)\,. \tag{11}$$

The following is the 'Lambert W function base s'[2], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A\,, \tag{12}$$

which looks very similar to (8). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s}\,. \tag{13}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A)\,, \tag{14}$$

[2]This notation I invented myself.

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma)e^{W_n(\gamma)}\,. \tag{15}$$