Math Diversion Problem 823

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It should be clear that there is no real content to
these proofs: all one has to do to obtain
a proof is to keep from getting confused.
— Robert Geroch[1]

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Source: Conceptual Mathematics: a first introduction
      to categories, 2nd Ed
Title: Isomorphims, Exercise 3a
  Presenter:  F. W. Lawvere & S. H. Schanuel
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1 Problem 3a

Function $f : B \to C$ has an inverse. Show that if

$$f \circ h = f \circ k\,, \tag{1}$$

then

$$h = k\,. \tag{2}$$

2 Solution

Let's begin by drawing a figure, where A, B, C are sets.

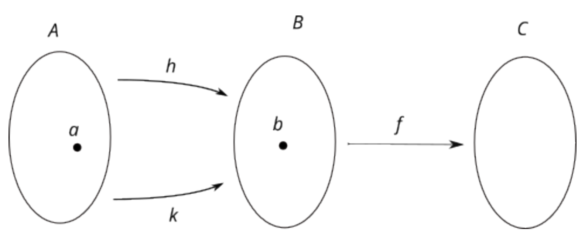

Figure 1. f^{-1} exists and $f \circ h = f \circ k$, where $h, k : A \to B$.

[1] R. Geroch, *Mathematical Physics*, Chicago Lectures in Physics, University Chicago Press, 1985, p. 6. (From an introductory chapter on category theory.)

What does it mean that $h = k$? It means that for every $a \in A$

$$h(a) = k(a)\,. \tag{3}$$

So what do we have to work with? From (1), we have that

$$(f \circ h)(a) = (f \circ k)(a) \quad \forall\, a \in A\,, \tag{4}$$

which means that

$$f(h(a)) = f(k(a)) \quad \forall\, a \in A\,. \tag{5}$$

But since f has an inverse, we can apply it to both sides of this last equation (on the left), to get

$$f^{-1}(f(h(a))) = f^{-1}(f(k(a))) \quad \forall\, a \in A\,. \tag{6}$$

But the f^{-1} and the f cancel, leaving us with

$$h(a) = k(a) \quad \forall\, a \in A\,, \tag{7}$$

which is exactly what we were to show. Hence

$$h = k\,. \tag{8}$$