Math Diversion Problem 825

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October 6, 2025

The truth is only effective against an
honest and sound mind.
— The Author

`Source:https://www.youtube.com/watch?v=I063-OrNNlY`
`Title: A Viewer Suggested Equation | Problem 293`
`Presenter: aplusbi`

1 Problem

Given the relation

$$z^{\frac{1}{z}} = i\,, \tag{1}$$

solve for z.

Note: WolframAlpha uses the Lambert W_n function but provides solutions only for values $n = -1, 0, 1$.

2 Solution

Rewrite (1) into the form

$$z^{z^{-1}} = i = e^{i\pi/2}e^{2i\pi m}\,, \tag{2}$$

where $i = e^{i\pi/2}$ and $e^{2i\pi m} = 1$ for all $m \in \mathbb{Z}$. Then take the natural logarithm, to get

$$z^{-1}\ln z = i\pi/2 + 2i\pi m = i\pi(\tfrac{1}{2} + 2m)\,. \tag{3}$$

Next, multiply through by -1 and use the rule $-\ln z = \ln z^{-1}$:

$$z^{-1}\ln z^{-1} = -i\pi(\tfrac{1}{2} + 2m)\,. \tag{4}$$

Now take the Lambert W function across this to get

$$\ln z^{-1} = W_n(-i\pi(\tfrac{1}{2} + 2m))\,. \tag{5}$$

Then raise e to the power of this last equation, to get

$$z^{-1} = e^{W_n(-i\pi(\frac{1}{2}+2m))} \,. \tag{6}$$

Lastly, invert:

$$z = e^{-W_n(-i\pi(\frac{1}{2}+2m))} \,. \tag{7}$$

And as I already said, WolframAlpha provides solutions only for values $n = -1, 0, 1$.

3 Appendix: Lambert W Function

Warning: The Lambert W function can be tricky because it's subtle. Please do not attempt to land any craft on Mars based on my calculations using the Lambert W function. We could both end up with Martian egg on our faces.

The Lambert W function goes as follows: If

$$ze^z = B \,, \tag{8}$$

then

$$z = W(B) \,, \tag{9}$$

where there are domain constraints on B that we won't go into here. Warning: This can be a complicated (multi-valued) function to deal with.

A lemma I'll need from the theory of the Lambert W function is the following: If

$$y \ln y = B \,, \tag{10}$$

then

$$\ln y = W(y \ln y) = W(B) \,. \tag{11}$$

The following is the 'Lambert W function base s'[1], or W_s, where s is a positive real number. Let's begin with the relation

$$xs^x = A \,, \tag{12}$$

which looks very similar to (8). Then

$$x = W_s(xs^x) \equiv \frac{W(A \ln s)}{\ln s} \,. \tag{13}$$

But when $s = e$, we have that

$$x = W_e(xe^x) = \frac{W(A \ln e)}{\ln e} = W(A) \,, \tag{14}$$

[1] This notation I invented myself.

which is the usual Lambert W function. (By the way, the proof to this lemma is not hard. It begins with setting $s^x = e^y$ and proceeding from there.)

If s is an integer, I may resort to putting parentheses around it to distinguish it from the n-series, as such $W_{(s)}$.

One last result we might need is

$$\gamma = W_n(\gamma) e^{W_n(\gamma)} . \tag{15}$$