Math Diversion Problem 831

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If others would think about mathematical truths as
deeply and as continuously as I have, they
would make my discoveries.
— Carl Friedrich Gauss

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Source: https://www.youtube.com/watch?v
             =2iBNo4j3vRo&list=PL3E4136E122545FBE
Title: Gamma Function - Part 2 - Functional Equation
  Presenter:  MrYouMath
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1 Introduction

This is the second part of a 12-part series on the Gamma function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 The Gamma Function – Part 2

The Gauss Representation

We first need a lemma:

$$e^{-t} = \lim_{n\to\infty} \left(1 + \frac{-t}{n}\right)^n . \tag{1}$$

Then

$$\begin{aligned} \Gamma(s) &= \int_0^\infty t^{s-1} \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n dt \\ &= \lim_{n\to\infty} \int_0^\infty t^{s-1} \left(1 - \frac{t}{n}\right)^n dt \,. \end{aligned} \tag{2}$$

Let's see what happens if we try an integration by parts (repeatedly).

$$
\begin{aligned}
\int_0^n t^{s-1}\left(1+\frac{-t}{n}\right)^n dt &= \cancelto{0}{\frac{1}{s}t^s\left(1-\frac{t}{n}\right)^n\Bigg|_{t=0}^{n}} + \int_0^n \frac{1}{s}t^s\left(1+\frac{-t}{n}\right)^{n-1}\left(-\frac{1}{n}\right) dt \\
&= \frac{n}{sn}\int_0^n \frac{1}{s}t^s\left(1+\frac{-t}{n}\right)^{n-1}\left(-\frac{1}{n}\right) dt \\
&= \frac{n}{sn}\frac{n-1}{(s+1)n}\int_0^n \frac{1}{s}t^{s+1}\left(1-\frac{t}{n}\right)^{n-2} dt \\
&= \cdots \\
&= \frac{n}{sn}\frac{n-1}{(s+1)n}\frac{n-2}{(s+2)n}\cdots\frac{1}{(s+n-1)n}\int_0^n \frac{1}{s}t^{s+n-1}dt \\
&= \frac{n}{sn}\frac{n-1}{(s+1)n}\frac{n-2}{(s+2)n}\cdots\frac{1}{(s+n-1)n}\frac{t^{s+n}}{s+n}\Bigg|_0^n . \qquad (3)
\end{aligned}
$$

Or,

$$
\int_0^n t^{s-1}\left(1+\frac{-t}{n}\right)^n dt = \frac{n!}{n^n}n^{s+n}\prod_{i=0}^{n}(s+i)^{-1} = \frac{n!n^s}{\prod_{i=0}^{n}(s+i)} . \qquad (4)
$$

Therefore, after separating out the $i=0$ factor of the product, we have that

$$
\Gamma(s) = \lim_{n\to\infty}\left[\frac{n^s}{s}\prod_{i=1}^{n}\frac{i}{(s+i)}\right], \qquad (5)
$$

and the $n!$ was replaced by i in the numerator of the product.