Math Diversion Problem 832

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I take the positivist viewpoint that a physical theory is just
a mathematical model and that it is meaningless to ask
whether it corresponds to reality. All that one can ask
is that its predictions should be in agreement
with observation. — Stephen Hawking
[*The Nature of Space and Time*, p.3-4]

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Source: https://www.youtube.com/watch?v=LHyg9-fBv_U
Title: A Trigonometric Exponential Equation | Problem #103
  Presenter:  aplusbi
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1 Problem

Given the relation

$$\cos\left(e^z\right) = i\,, \tag{1}$$

solve for z.

2 Preparation

$$\cos\phi = \frac{e^{i\phi} + e^{-i\phi}}{2}\,. \tag{2}$$

Given $e^z = A$, we take the natural logarithm to solve for z, but then we get

$$z = \ln A + 2\pi n,\;\; n \in \mathbb{Z}\,. \tag{3}$$

To see why this is true, we just reverse it!

$$\begin{aligned} e^z &= e^{\ln A + 2\pi n},\;\; n \in \mathbb{Z} \\ &= e^{\ln A} e^{2\pi n},\;\; n \in \mathbb{Z} \\ &= e^{\ln A} = A\,, \end{aligned} \tag{4}$$

because

$$e^{2\pi n} = 1 \tag{5}$$

for all integers n.

3 Solution

We begin by rewriting (1) to

$$\cos\left(e^{z}\right) = \frac{e^{ie^{z}} + e^{-ie^{z}}}{2} = i\,, \tag{6}$$

which can be rewritten as

$$e^{2ie^{z}} + 1 = 2i e^{ie^{z}}\,. \tag{7}$$

After a little algebra, we can rewrite this into a quadratic form:

$$\left(e^{ie^{z}}\right)^{2} - 2i e^{ie^{z}} + 1 = 0\,. \tag{8}$$

This has the solution for $e^{ie^{z}}$ in the form

$$e^{ie^{z}} = \frac{2i \pm \sqrt{-4 - 4(1)(1)}}{2} = i \pm i\sqrt{2}\,. \tag{9}$$

On taking the natural logarithm across this, we get

$$ie^{z} = \ln\left[i \pm i\sqrt{2}\,\right]. \tag{10}$$

Lastly, multiply through by $-i$ and then take one more logarithm and we have that

$$z = \ln\left\{-\,i\ln\left[\,i \pm i\sqrt{2}\,\right]\right\} + 2\pi n,\ \ n \in \mathbb{Z}\,. \tag{11}$$