Math Diversion Problem 833

P. Reany

October 10, 2025

Theory like mist on eyeglasses — obscures facts.
— Charlie Chan

```
Source: https://www.youtube.com/watch?v
             =2iBNo4j3vRo&list=PL3E4136E122545FBE
Title: Gamma Function - Part 3 - Weierstrass Representation
  Presenter:  MrYouMath
```

1 Introduction

This is the third part of a 12-part series on the Gamma function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 Weierstrass Representation – Part 3

The Weierstrass Representation of the Gamma function is

$$\Gamma(s) = \frac{1}{s} e^{-\gamma s} \prod_{i=1}^{\infty} e^{\frac{s}{i}} \left(1 + \frac{s}{i}\right)^{-1}, \tag{1}$$

where γ is the Euler-Macheroni constant, defined by

$$\gamma \equiv \lim_{n\to\infty} \left(\sum_{i=1}^{n} \frac{1}{i} - \log n \right), \tag{2}$$

To prove this, we begin with the Gauss representation and massage it. Note that $e^0 = 1$.[1]

$$\Gamma(s) = \lim_{n\to\infty}\left[\frac{n^s}{s}\prod_{i=1}^{n}\frac{i}{(s+i)}\right]$$
$$= \lim_{n\to\infty}\left[\frac{n^s}{s}\prod_{i=1}^{n}\left(1+\frac{s}{i}\right)^{-1}\right]$$
$$= \lim_{n\to\infty}\left[\frac{1}{s}e^0 e^{s\log n}\prod_{i=1}^{n}\left(1+\frac{s}{i}\right)^{-1}\right]$$
$$= \lim_{n\to\infty}\left[\frac{1}{s}e^{\left(\sum_{i=1}^{n}\frac{s}{i}-\sum_{i=1}^{n}\frac{s}{i}\right)+s\log n}\prod_{i=1}^{n}\left(1+\frac{s}{i}\right)^{-1}\right]$$
$$. \tag{3}$$

Or,

$$\Gamma(s) = \lim_{n\to\infty}\left[\frac{1}{s}e^{\left(\sum_{i=1}^{n}\frac{s}{i}-\gamma s\right)}\prod_{i=1}^{n}\left(1+\frac{s}{i}\right)^{-1}\right]$$
$$= \lim_{n\to\infty}\left[\frac{1}{s}e^{-\gamma s}\prod_{i=1}^{n}e^{s/i}\left(1+\frac{s}{i}\right)^{-1}\right]$$
$$, \tag{4}$$

where we used that, for the sum on i:

$$e^{\sum \frac{s}{i}} = \prod e^{\frac{s}{i}}\,. \tag{5}$$

[1]We will use the trick that $n^s = e^{s\log n}$.