Math Diversion Problem 837

P. Reany

October 12, 2025

The definition of a good mathematical problem
is the mathematics it generates
— Andrew Wiles

```
Source: https://www.youtube.com/watch?v
             =2iBNo4j3vRo&list=PL3E4136E122545FBE
Title: Gamma Function - Part 5 - Gamma of 1/2
  Presenter:  MrYouMath
```

1 Introduction

This is the fifth part of a 12-part series on the Gamma function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

Note:

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s} \,. \tag{1}$$

2 Gamma of 1/2

Referencing the last equation (1) and substituting into it $s = \frac{1}{2}$, we get

$$\big[\Gamma(\tfrac{1}{2})\big]^2 = \pi \,, \tag{2}$$

from we we get the simpler expression

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi} \,. \tag{3}$$

Okay, let's leverage what we've got:

$$\Gamma(\tfrac{1}{2}) = \int_0^\infty t^{\frac{1}{2}-1} e^t \, dt = \int_0^\infty \frac{1}{\sqrt{t}} e^t \, dt \,. \tag{4}$$

Therefore,

$$\int_0^\infty \frac{1}{\sqrt{t}}\, e^{-t}\, dt = \sqrt{\pi}\,. \tag{5}$$

So, what else can we get out of this? Let $t \to pu^2$ with $p > 0$:

$$2p \int_0^\infty e^{-pu^2}\, dt = \sqrt{\pi}\,. \tag{6}$$

From this we get the more symmetric integral

$$\int_{-\infty}^\infty e^{-pu^2}\, dt = \sqrt{\pi/p}\,. \tag{7}$$

This is the so-called Gaussian Integral.