Math Diversion Problem 839

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> The human mind has never invented a labor-saving machine equal to algebra.
> — J. Willard Gibbs

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Source: https://www.youtube.com/watch?v
             =2iBNo4j3vRo&list=PL3E4136E122545FBE
Title: Gamma Function - Part 6 - Stirling's Formula
  Presenter:  MrYouMath
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1 Introduction

This is the sixth part of a 12-part series on the Gamma function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 The Stirling Approximation – Part 6

The following is the The Stirling Approximation to $n!$:

$$n! \approx \frac{n^n}{e^n}\sqrt{2n\pi}\,. \tag{1}$$

We will build up this result through a long process. We begin with

$$\log\left(t^n e^{-t}\right) = n\log t - t\,. \tag{2}$$

Now, let $t \to n+\epsilon$:

$$n\log t - t = n\log\left(n+\epsilon\right) - \left(n+\epsilon\right). \tag{3}$$

But

$$\log\left(n+\epsilon\right) = \log n(1+\epsilon/n) = \log n + \log\left(1+\epsilon/n\right). \tag{4}$$

Now, for $n >> 1$, ϵ/n is small, hence

$$\log\left(1+\epsilon/n\right) = \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\frac{\epsilon^k}{n^k} \tag{5}$$

can be approximated by just a few terms. So

$$
\begin{aligned}
n \log{(n+\epsilon)} - (n+\epsilon) &= n\left(\log n + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{\epsilon^k}{n^k}\right) - n - \epsilon \\
&= n \log n - n + \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{\epsilon^k}{n^{k-1}} - \epsilon \\
&= n \log n - n - \frac{\epsilon^2}{2n} + \frac{\epsilon^3}{3n^2} - \frac{\epsilon^4}{4n^3} + \cdots . \qquad (6)
\end{aligned}
$$

Also

$$\log{(t^n e^{-t})} \approx n \log n - n - \frac{\epsilon^2}{2n} . \qquad (7)$$

Therefore,

$$t^n e^{-t} \approx \frac{n^n}{e^n} e^{-\epsilon^2/2n} . \qquad (8)$$

Thus,

$$n! = \int_0^\infty t^n e^{-t} dt \approx \int_{-n}^\infty \frac{n^n}{e^n} e^{-\epsilon^2/2n} d\epsilon . \qquad (9)$$

And we see the Gaussian integral in variable ϵ. So,

$$n! \approx \frac{n^n}{e^n} \sqrt{2n\pi} . \qquad (10)$$