Math Diversion Problem 841

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You will never plough a field if you only
turn it over in your mind.
— Irish Proverb

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Source: https://www.youtube.com/watch?v
             =2iBNo4j3vRo&list=PL3E4136E122545FBE
Title: Gamma Function - Part 7 - The Euler Integral I
  Presenter:  MrYouMath
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1 Introduction

This is the seventh part of a 12-part series on the Gamma function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 The Euler Integral I – Part 7

Once again we begin with

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt \,, \tag{1}$$

and make the substitution $t \to pu^n$ where p is complex. Then

$$\Gamma(s) = \int_0^\infty np^s u^{ns-1} e^{-pu^n} du \,, \tag{2}$$

which can be put into the alternative form

$$\frac{\Gamma(s)}{np^s} = \int_0^\infty u^{ns-1} e^{-pu^n} du \,. \tag{3}$$

Okay, we will denote the complex conjugate by an overbar. In the last equation we can replace p by its complex conjugate, to get[1]

$$\frac{\Gamma(s)}{n\bar{p}^{\,s}} = \int_0^\infty u^{ns-1} e^{-\bar{p}\, u^n} du \,. \tag{4}$$

[1] We can do this because p was an arbitrary complex number to begin with.

On taking a combination of the last two equations, we get

$$\frac{\Gamma(s)}{np^s} \pm \frac{\Gamma(s)}{n\bar{p}^{\,s}} = \int_0^\infty u^{ns-1}\, e^{-pu^n}\, du \pm \int_0^\infty u^{ns-1}\, e^{-\bar{p}\,u^n}\, du\,. \tag{5}$$

Now, if we write p in the polar form, we get

$$p = |\,p\,|\, e^{i\alpha} = a + ib\,. \tag{6}$$

With this result, (5) becomes

$$\frac{\Gamma(s)}{n\,|\,p\,|^s}\left[\frac{1}{e^{-ias}} \pm \frac{1}{e^{i\alpha s}}\right] = \int_0^\infty u^{ns-1}\, e^{-\alpha u^n}\big(e^{isu^n} \pm e^{-isu^n}\big) du\,. \tag{7}$$

By taking first the + sign and then the − sign, and stacking them:

$$\frac{\Gamma(s)}{n\,|\,p\,|^s}\begin{bmatrix} 2\cos{(\alpha s)} \\ 2\sin{(\alpha s)}\end{bmatrix} = \int_0^\infty u^{ns-1}\, e^{-au^n}\begin{bmatrix}\cos{(bu^u)} \\ \sin{(bu^n)}\end{bmatrix} du\,, \tag{8}$$

where

$$\begin{cases}\tan\alpha = b/a & a \neq 0\,,\\ \alpha = \pi/2 & a = 0\,.\end{cases} \tag{9}$$