

Math Diversion Problem 846

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Abstract

The Cauchy-Riemann Equations.

You are *not* the technology you have access to.
You *are* what you carry around inside of you —
intellectually, emotionally, and spiritually.
— The Author

1 Introduction

Our task here is to determine a sufficiency condition on functions from the complex plane of variables $z = x + iy$ to the complex plane $w = u + iv$, where u and v are real functions of real variables x and y .

2 Problem

Derive the Cauchy-Riemann equations.

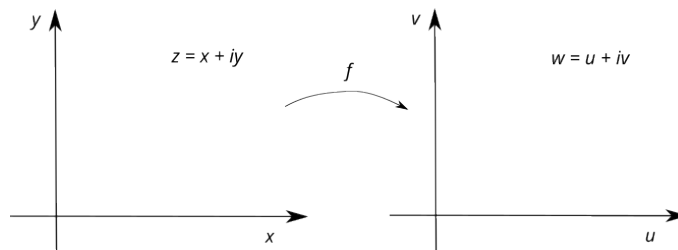


Figure 1. Complex variable mapping: $f = w(z)$.

When we defined the differentiation of functions over a single real variable (via limits), we demanded that the derivative at a point be the same as the

limit approached from the left side and from the right side. This required the functions to be smooth over some open interval containing the point.

But when we move to notions of differentiating complex functions over the complex plane, we need more requirements, such as the partial derivatives of u and v to exist and be continuous and the values of $f'(z)$ at point z to be independent of the direction by which the limit is taken. It won't be proved here, but it will suffice to find conditions by which $f'(z)$ has the same value when approached from any two linearly independent directions in the z -plane. And since we are free to choose these two directions, we choose the horizontal and vertical directions.

In general, for $f'(z)$ to exist, then on some open domain \mathcal{D} in the z -plane,

$$f'(z) = \lim_{\Delta z \rightarrow 0} \frac{w(z + \Delta z) - w(z)}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}. \quad (1)$$

Now, we take this limit along a horizontal path (a line parallel to the x axis, hence $\Delta y = 0$), to get

$$\begin{aligned} f'(z) \Big|_{\substack{x \text{ direction} \\ \Delta y = 0}} &= \lim_{\Delta x \rightarrow 0} \frac{\Delta u + i\Delta v}{\Delta x} = \lim_{\Delta x \rightarrow 0} \frac{\Delta u}{\Delta x} + i \lim_{\Delta x \rightarrow 0} \frac{\Delta v}{\Delta x} \\ &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}. \end{aligned} \quad (2)$$

Similarly, when we take the limit along a vertical path, we have that

$$\begin{aligned} f'(z) \Big|_{\substack{y \text{ direction} \\ \Delta x = 0}} &= \lim_{\Delta y \rightarrow 0} \frac{\Delta u + i\Delta v}{i\Delta y} = (-i) \lim_{\Delta y \rightarrow 0} \frac{\Delta u}{\Delta y} + \lim_{\Delta y \rightarrow 0} \frac{\Delta v}{\Delta y} \\ &= -i \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}. \end{aligned} \quad (3)$$

And since we have demanded that these two limits in the last two equations must be the same, then we must set their respective real and imaginary parts equal to each other, as in

$$\text{Real part : } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad (4a)$$

$$\text{Imaginary part : } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \quad (4b)$$

and these are the Cauchy-Riemann Equations.

3 Add in Analyticity

Now, if we want to have complex differentiation really powerful (no pun intended), we need to make the derivative analytic. We do this by insisting that in addition to our requiring the Cauchy-Riemann equations to hold, we also

require that the four partial derivatives of u and v (i.e, u_x, u_y, v_x, v_y) not only to exist but also to be continuous on the domain \mathcal{D} .

By doing this we get a number of benefits: We get that the mixed partial derivatives of u and v commute (Clairaut's theorem), and that $f(z)$ is infinitely differentiable, and that $f(z)$ can be expressed in power series form.

Thus: A function f is **analytic** on a region $D \subset \mathbb{C}$ if it's complex differentiable at every point in D .

In complex number theory, analytic and holomorphic are equivalent terms. A function is considered **analytic** if it has a convergent Taylor Series on some open disk centered at that point, and it is **holomorphic** if it is differentiable on some open disk centered at that point. Both definitions imply the other, making them interchangeable in the context of complex functions. This equivalence is a fundamental theorem in complex analysis, highlighting the deep connection between these two properties of complex functions. [— Copilot]

4 Variables u and v are Harmonic

Differentiating (4a) and (4b) and using the fact that the partial derivatives commute, yields,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y}, \quad \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial x \partial y}, \quad (5)$$

together with similar calculations for the derivatives of v , we get,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \quad (6)$$

We can write these last equations equivalently as

$$\nabla^2 u = 0, \quad \nabla^2 v = 0. \quad (7)$$

These are Laplace's equations, and it makes u and v as **harmonic** functions.