Math Diversion Problem 847

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The essence of mathematics is not to make simple things
complicated, but to make complicated things simple.
— S. Gudder

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Source: https://www.youtube.com/watch?v
          =2iBNo4j3vRo&list=PL3E4136E122545FBE
Title: Gamma Function - Part 10 - The Beta Function
  Presenter:  MrYouMath
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1 Introduction

This is the tenth part of a 12-part series on the Gamma function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 The Beta Function – Part 10

The Beta function is defined by

$$B(s_1, s_2) \equiv \int_0^1 u^{s_1-1}(1-u)^{s_2-1} du \,. \tag{1}$$

We begin with the exploratory product of two independent Gamma functions:

$$\Gamma(s_1)\Gamma(s_2) = \int_0^\infty x^{s_1-1} e^{-x}\, dx \int_0^\infty y^{s_2-1} e^{-y}\, dy \,. \tag{2}$$

Next, a change of variables: $x \to u^2$ and $y \to v^2$. Then

$$\begin{aligned}
\Gamma(s_1)\Gamma(s_2) &= \int_0^\infty u^{2s_1-2} e^{-u^2}\, 2u\, du \int_0^\infty v^{2s_2-2} e^{-v^2}\, 2v\, dv \\
&= 4\int_0^\infty u^{2s_1-1} e^{-u^2}\, du \int_0^\infty v^{2s_2-1} e^{-v^2}\, dv \\
&= 4\int_0^\infty \left[\int_0^\infty v^{2s_2-1} e^{-v^2}\, dv\right] u^{2s_1-1} e^{-u^2}\, du \\
&= 4\int_0^\infty \int_0^\infty v^{2s_2-1} e^{-v^2}\, u^{2s_1-1} e^{-u^2}\, dv\, du \\
&= 4\int_0^\infty \int_0^\infty v^{2s_2-1}\, u^{2s_1-1} e^{-(u^2+v^2)}\, dv\, du \\
&.
\end{aligned} \tag{3}$$

Next, one more variable substitution: $v = r\cos\alpha$, $u = r\sin\alpha$, yielding

$$\begin{aligned}
\Gamma(s_1)\Gamma(s_2) &= 4\int_0^\infty \int_0^{\pi/2} r^{2s_2-1} (\cos\alpha)^{2s_2-1} r^{2s_1-1} (\sin\alpha)^{2s_1-1} e^{-r^2}\, r\, d\alpha\, dr \\
&= 4\int_0^\infty \int_0^{\pi/2} r^{2s_2-1} r^{2s_1-1} r (\cos\alpha)^{2s_2-1} (\sin\alpha)^{2s_1-1} e^{-r^2}\, d\alpha\, dr \\
&= 4\int_0^\infty \left[\int_0^{\pi/2} (\cos\alpha)^{2s_2-1} (\sin\alpha)^{2s_1-1} d\alpha\right] r^{2s_2-1+2s_1-2} e^{-r^2}\, r\, dr \\
&= \int_0^\infty (r^2)^{s_1+s_2-1} e^{-r^2}\, 2r\, dr \times \int_0^{\pi/2} 2(\cos\alpha)^{2s_2-1} (\sin\alpha)^{2s_1-1} d\alpha\,.
\end{aligned} \tag{4}$$

From this we get the simpler form

$$\Gamma(s_1)\Gamma(s_2) = \Gamma(s_1+s_2)\ 2\int_0^{\pi/2} (\cos\alpha)^{2s_2-1} (\sin\alpha)^{2s_1-1} d\alpha\,. \tag{5}$$

Hence,

$$\begin{aligned}
\frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} &= 2\int_0^{\pi/2} (\cos\alpha)^{2s_2-1} (\sin\alpha)^{2s_1-1} d\alpha \\
&= \int_0^{\pi/2} (\cos\alpha)^{2s_2-2} (\sin\alpha)^{2s_1-2} (2\sin\alpha\cos\alpha)\, d\alpha\,.
\end{aligned} \tag{6}$$

And one last substitution: Let $u = (\sin\alpha)^2$, then

$$B(s_1,s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1+s_2)} = \int_0^1 u^{s_1-1}(1-u)^{s_2-1}\, du\,. \tag{7}$$