Math Diversion Problem 849

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Lifelong learning is no longer a luxury but
a necessity for employment.
— Jay Samit

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Source: https://www.youtube.com/watch?v
          =2iBNo4j3vRo&list=PL3E4136E122545FBE
Title: Gamma Function - Part 11 - The Legendre Duplication Formula
  Presenter:  MrYouMath
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1 Introduction

This is the eleventh part of a 12-part series on the Gamma function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 Preparation

$$B(s_1, s_2) = \frac{\Gamma(s_1)\Gamma(s_2)}{\Gamma(s_1 + s_2)} = \int_0^1 u^{s_1 - 1}(1-u)^{s_2 - 1}\, du\,. \tag{1}$$

3 The Legendre Duplication Formula

This formula looks like the following

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\,\Gamma(s)\Gamma(s + \tfrac{1}{2})\,. \tag{2}$$

We start with Eq. (1) and set $s_1 = s_2 = s$ to get

$$\frac{\Gamma(s)\Gamma(s)}{\Gamma(2s)} = \int_0^1 u^{s-1}(1-u)^{s-1}\, du\,. \tag{3}$$

Not let's make the following variable substitution to the integral on the RHS of this last equation, namely, $u \to \frac{1+x}{2}$, then

$$\begin{aligned}\text{RHS} &= \int_0^1 \Big(\frac{1+x}{2}\Big)^{s-1}\Big(\frac{1-x}{2}\Big)^{s-1}\frac{dx}{2} \\ &= \frac{1}{2^{2s-1}}\int_{-1}^{1}(1-x^2)^{s-1}\,dx \\ &= \frac{1}{2^{2s}}\int_0^1 (1-x^2)^{s-1}\,dx\,. \end{aligned} \tag{4}$$

But, using the Beta Function, we have that

$$B(\tfrac{1}{2}, s) = 2\int_0^1 (1-x^2)^{s-1}\,dx\,. \tag{5}$$

Hence, from (1)

$$2^{2s-1}\Gamma(s)\Gamma(s) = \Gamma(2s)B(\tfrac{1}{2}, s) = \frac{\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(\frac{1}{2}+s)}\,. \tag{6}$$

With some manipulation we get

$$2^{2s-1}\Gamma(s)\Gamma(s)\Gamma(\tfrac{1}{2}+s) = \sqrt{\pi}\,\Gamma(2s)\,. \tag{7}$$

And, on solving for $\Gamma(2s)$, we have that

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma(\tfrac{1}{2}+s)\,. \tag{8}$$