Math Diversion Problem 851

P. Reany

October 19, 2025

The most dangerous phrase in the language is,
'We've always done it this way.'
— Grace Hopper, computer pioneer

```
Source: https://www.youtube.com/watch?v
          =2iBNo4j3vRo&list=PL3E4136E122545FBE
Title: Gamma Function - Part 12 - Relation to the Zeta Function
  Presenter:  MrYouMath
```

1 Introduction

This is the twelfth part of a 12-part series on the Gamma function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 Relation to the Zeta Function – Part 12

Let s be a complex variable. Then the Gamma function of s is given as

$$\Gamma(s) \equiv \int_0^\infty t^{s-1} e^{-t} dt \quad \text{where } \mathrm{Re}(s) > 0\,. \tag{1}$$

What we're about to do is to show a relationship between the Gamma function and the Zeta function. So, let's begin by making a variable substitution in (2), $t \to nu$:

$$\begin{aligned}\Gamma(s) &= \int_0^\infty (nu)^{s-1} e^{-nu} n\,du \\ &= \int_0^\infty n^s u^{s-1} e^{-nu}\, du\,. \end{aligned} \tag{2}$$

Hence,

$$\Gamma(s)\frac{1}{n^s} = \int_0^\infty u^{s-1} e^{-nu}\, du\,. \tag{3}$$

Does the fraction in the LHS look suggestive in the present context? Either way, let's perform a summation on both sides by n, with it going from 1 to infinity:

$$\Gamma(s) \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} u^{s-1} \sum_{n=1}^{\infty} e^{-nu}\, du\,. \tag{4}$$

By substituting the zeta function on the left and using the geometric series trick on the right, we get

$$\Gamma(s)\zeta(s) = \int_0^{\infty} u^{s-1} \left[\frac{1}{1-e^{-u}} - 1\right] du = \int_0^{\infty} u^{s-1} \frac{du}{e^u - 1}\,. \tag{5}$$