Math Diversion Problem 853

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If you misjudge the actual nature of the problem that
besets you, you'll likely also misjudge the solution
to it, making things even worse.
— The Author

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Source: https://www.youtube.com/watch?v=ZlYfEqdlhk0
Title: Zeta Function - Part 1 -  Convergence
  Presenter:  MrYouMath
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1 Introduction

This is the first part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 Preparation

This series of papers will focus on some of the best known and useful functions of analytic number theory, such as the zeta function, the beta function, the gamma function, and many more. My interest in them originally came from the fact that these functions have applications in physics. But I have come to find interest in them for the shear beauty of their theorems, proofs, and wide applicability.

Classically, analytic number theory has its reason for being in the study of the integers, and the study of the integers is centered on the study of primes. And, although we will begin this series with some dazzling results that use the prime numbers, we won't linger on the prime numbers for long.

Some background material on complex numbers is needed. Consider the complex function e^{ix}. Whatever the value of the real number x, the magnitude of e^{ix} is unity, or 1.

$$\left| e^{ix} \right| = 1 \,. \tag{1}$$

For a given x value, we can think of e^{ix} as a point of the unit circle in the complex plane.

Another property of complex exponentials we need to know is the simple rule for complex numbers A and B:

$$e^{A+B} = e^{A}e^{B}\,. \tag{2}$$

This simple relation holds true only because the complex numbers is a commutative set. Anyway, if we set $A = x$ and $B = iy$ (where x and y are real numbers), we have that

$$e^{x+iy} = e^{x}e^{iy}\,. \tag{3}$$

Next, we need to know how to take the absolute value of a product of complex numbers, which is as follows.

$$|\,AB\,| = |\,A\,|\,|\,B\,|\ , \tag{4}$$

for complex numbers A and B. Applying this form to (3) and using (1), we get

$$\left|\,e^{x+iy}\,\right| = |\,e^{x}\,|\left|\,e^{iy}\,\right| = |\,e^{x}\,|\ . \tag{5}$$

So, here's where we begin with the first video of MrYouMath:

3 Convergence the Zeta Function

We return now the Euler Zeta function:

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots\,. \tag{6}$$

The method we will use to decide if the Euler Zeta function is convergent, or, more properly, where if anywhere it is convergent, we'll use the well-know notion of absolute convergence. Consider the following series

$$S = \sum_{n=1}^{\infty} s_n\,. \tag{7}$$

S is said to "converge absolutely" if the following sum converges, i.e., for some positive real number L,[1]

$$\sum_{n=1}^{\infty}|\,s_n\,| = L\,. \tag{8}$$

Thus, we write

$$|\,\zeta(s)\,| \leq \sum_{n=1}^{\infty}\left|\,\frac{1}{n^s}\,\right| = \sum_{n=1}^{\infty}\frac{1}{|\,n^s\,|} = \sum_{n=1}^{\infty}\frac{1}{|\,n^{x+iy}\,|} = \sum_{n=1}^{\infty}\frac{1}{|\,n^{x}\,|\,|\,n^{iy}\,|}\,. \tag{9}$$

[1] And we have already started to use the techniques of 'analysis' into our discussion.

But, look what we can do now:

$$n^{iy} = e^{iy \ln n}\,, \tag{10}$$

which is pretty sneaky, if you ask me. Anyway,

$$\left|\, n^{iy} \,\right| = \left|\, e^{iy \ln n} \,\right| = 1\,. \tag{11}$$

Hence, (9) becomes

$$|\,\zeta(s)\,| \leq \sum_{n=1}^{\infty} \frac{1}{|\,n^x\,|} = \sum_{n=1}^{\infty} \frac{1}{n^x}\,, \tag{12}$$

which will converge when $x > 1$.