Math Diversion Problem 854

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Learning is a treasure that will follow its owner everywhere.
— Chinese proverb

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Source: YouTube
Title: A Log-Lambert Problem
  Presenter:  Math Olympiad
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1 Problem

Given the relation

$$3^k = k^9\,, \tag{1}$$

find the solutions of k.

2 Solution

Part #1:

Let $k = 3^\alpha$, then we have that

$$3^{3^\alpha} = 3^{9\alpha}\,. \tag{2}$$

On setting the exponents equal, we get

$$3^\alpha = 9\alpha = 3^2\alpha\,. \tag{3}$$

After a little algebra, we have that

$$3^{\alpha-2} = \alpha\,. \tag{4}$$

This equation can be solved by inspection to get

$$\alpha = 3\,. \tag{5}$$

Therefore

$$k = 3^3 = 27\,. \tag{6}$$

Part #2:

To look for solutions other than just integers, we proceed as follows:

Start by taking the natural logarithm across (1), to get

$$k \ln 3 = 9 \ln k \,, \tag{7}$$

which, after a little algebra, becomes

$$\frac{1}{9} \ln 3 = k^{-1} \ln k \,. \tag{8}$$

Now we multiply through by -1:

$$-\frac{1}{9} \ln 3 = -k^{-1} \ln k = k^{-1} \ln k^{-1} \,. \tag{9}$$

Next, we take the Lambert W function across this equation to get

$$W(-\frac{1}{9} \ln 3) = \ln k^{-1} \,. \tag{10}$$

Solving this for k^{-1}, we have that

$$k^{-1} = e^{W(-\frac{1}{9} \ln 3)} \,. \tag{11}$$

If we wanted, we could solve for k by taking the multiplicative inverse, but I want the answer in the form that WolframAlpha poses it. So, I multiply through by $W(-\frac{1}{9} \ln 3)$, to get

$$W(-\frac{1}{9} \ln 3)\, k^{-1} = W(-\frac{1}{9} \ln 3)\, e^{W(-\frac{1}{9} \ln 3)} \,. \tag{12}$$

Next, we use the identity:

$$W(z)\, e^{W(z)} = z \,. \tag{13}$$

Then, (12) becomes

$$W(-\frac{1}{9} \ln 3)\, k^{-1} = -\frac{1}{9} \ln 3 \,. \tag{14}$$

After some algebra and generalizing the solution, we have that

$$k = -9 \frac{W_n(-\frac{1}{9} \ln 3)}{\ln 3} \quad \text{where } n \in \mathbb{Z} \,. \tag{15}$$