Math Diversion Problem 855

P. Reany

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Proofs are embedded in broader frameworks—moduli spaces,
categories, field theories—where the relationships
matter more than the isolated result.
— Copilot, being philosophical

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Source: https://www.youtube.com/watch?v=I3qSCWNXZKg
Title: Zeta Function - Part 2 -  Euler Product
  Presenter:  MrYouMath
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1 Introduction

This is the second part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 Euler Product – Part 2

Well, if you thought that was sneaky, what till you see the so-called 'Euler Product'. Not willing to be content to write his Euler zeta function as an infinite sum, Euler figured out how to write it equivalently as an infinite product, as follows

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - p^{-s}}\,, \tag{1}$$

where P stands for the set of all positive prime numbers. To appreciate the proof, we need to get used to the closed form for what's called the 'geometric sum'. So, let x be a complex number and let's take the sum and get

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}\,, \tag{2}$$

and, of course, there is a condition for the convergence of this sum, that being that $|\,x\,| < 1$.[1]

[1] The reason this sum is called 'geometric' is because the ratio of a term, say, x^{m+1}, to its preceeding term, x^m, is a fixed value, in this case being the number x.

To appreciate the RHS of (1), let's start by choosing an arbitrary prime q of the set P and write

$$\frac{1}{1-q^{-s}}\ , \tag{3}$$

Now, we fit this into (2) to get

$$\sum_{k=0}^{\infty}(q^{-s})^k = \frac{1}{1-q^{-s}}\ , \tag{4}$$

where we have changed the index of summation to k for convience later on. For the next step, let's index all the positive prime numbers, starting with $i = 1$ and let $q \to p_i$. Then, we can rewrite the last equation to get

$$\frac{1}{1-p_i^{-s}} = \sum_{k_i=0}^{\infty}(p_i^{-s})^k = \sum_{k_i=0}^{\infty} p_i^{-sk_i} = \sum_{k_i=0}^{\infty}\frac{1}{p_i^{sk_i}}\ , \tag{5}$$

where we have set up a summation index k_i for each prime p_i. Now, we're going back to the RHS of (1):

$$\begin{aligned}\prod_{p\in P}\frac{1}{1-p^{-s}} &= \prod_{i=1}^{\infty}\frac{1}{1-p_i^{-s}} = \prod_{i=1}^{\infty}\sum_{k_i=0}^{\infty}\frac{1}{p_i^{sk_i}} \\ &= \sum_{k_1=0}^{\infty}\frac{1}{p_1^{sk_1}}\cdot\sum_{k_2=0}^{\infty}\frac{1}{p_2^{sk_2}}\cdot\sum_{k_3=0}^{\infty}\frac{1}{p_3^{sk_3}}\cdots \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\cdots\left[\frac{1}{p_1^{sk_1}}\cdot\frac{1}{p_2^{sk_2}}\cdot\frac{1}{p_3^{sk_3}}\cdots\right] \\ &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\cdots\left[\frac{1}{p_1^{k_1}p_2^{k_2}p_3^{k_3}\cdots}\right]^s .\end{aligned} \tag{6}$$

Now, here is where the magic happens! Take close look at this infinite product.

$$p_1^{k_1}p_2^{k_2}p_3^{k_3}\cdots\ , \tag{7}$$

How do we get the number 1 out of this product? Easy. Just set all the k_i's to zero. To get all the prime numbers out of this, we just set all but one of the k_i's to zero. The one we don't set to zero, we set to unity. For an arbitrary composite number $N = \prod_{j=1}^{m} p_{k_j}$, we set to zero every k_i not represented in this product, and set the exponent to all other primes to the value k_j. Therefore, by this procedure, we can construct every natural number exactly once! Therefore, the RHS of (6) becomes

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\cdots\left[\frac{1}{p_1^{k_1}p_2^{k_2}p_3^{k_3}\cdots}\right]^s = \sum_{n=1}^{\infty}\left[\frac{1}{n}\right]^s = \sum_{n=1}^{\infty}\frac{1}{n^s}\ . \tag{8}$$

Hence we have that

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s} = \prod_{p\in P}\frac{1}{1-p^{-s}}\ . \tag{9}$$