Math Diversion Problem 857

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You cannot read mathematics the way you read a
novel. If you zip through a page in less than an
hour, you are probably going too fast.
— Sheldon Axler
(from *Linear Algebra Done Right*)

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Source: https://www.youtube.com/watch?v=TDdGisWD5OU
Title: Zeta Function - Part 3 -  Euler Product Revisited
  Presenter:  MrYouMath
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1 Introduction

This is the third part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 From earlier in the series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - p^{-s}} \,. \tag{1}$$

3 Euler Product Revisited:

So, we return to the Euler zeta function with the goal to reproduce (1) in another way.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \,. \tag{2}$$

If we multiply this through by $1/2^s$, we get

$$\zeta(s)\frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{n^s}\frac{1}{2^s} = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} \,. \tag{3}$$

Next, we subtract (3) from (2) to get

$$\zeta(s)\left(1-\frac{1}{2^s}\right)=\sum_{n=1}^{\infty}\frac{1}{n^s}-\sum_{n=1}^{\infty}\frac{1}{(2n)^s}=\sum_{\substack{n=1\\n\neq 2k}}^{\infty}\frac{1}{n^s}\,, \tag{4}$$

where by $n \neq 2k$ we mean that n may not take on an even value. So, if we employ the same trick for 3 in addition for 2, we get

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)=\sum_{\substack{n=1\\n\neq 2k\\n\neq 3k}}^{\infty}\frac{1}{n^s}\,, \tag{5}$$

If we imagine continuing this process for all primes, the RHS term will boil down to just the first term, where $n = 1$. Therefore,

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\cdots\left(1-\frac{1}{p_N^s}\right)\cdots=1\,, \tag{6}$$

where P_n is the Nth prime. And so what's left is an infinite-product form of the Euler Zeta function:

$$\zeta(s)=\prod_{p\in P}\frac{1}{1-p^{-s}}\,. \tag{7}$$