Math Diversion Problem 859

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Dear Algebra, stop asking us to find your X,
she's not coming back.
— Woody Paige

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Source:  https://www.youtube.com/watch?v=SKa7b-3C32A
Title: Zeta Function - Part 4 -  Infinitude of Primes
  Presenter:  MrYouMath
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1 Introduction

This is the fourth part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 From earlier in the series:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1 - p^{-s}} \,. \tag{1}$$

3 Infinitude of primes

As a corollary, we know that for $s = 1$ in (1), the sum does not converge, therefore the product also cannot converge. However, if there were a finite number of primes, the product

$$\prod_{p \in P} \frac{1}{1 - p^{-s}} \tag{2}$$

must converge! This forces us to conclude that there must be an infinite number of primes.