Math Diversion Problem 861

P. Reany

October 24, 2025

A conversation between a father and son:

"Can I have a motorcycle when I get old enough?"
"If you take care of it."
"What do you have to do?"
"Lot's of things. You've
been watching me."
"Will you show me all of them?"
"Sure."
"Is it hard?"
"Not if you have the right attitudes.
It's having the right
attitudes that's hard."
— Robert Pirsig to his son, from
Zen and the Art of
Motorcycle Maintenance

```
Source:   https://www.youtube.com/watch?v=3eN9tQX3JJ4
Title: Zeta Function - Part 5
          -  Divergence of reciprocal sum of Prime Numbers.
  Presenter:  MrYouMath
```

1 Introduction

This is the fifth part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 The Objective This Time:

Show that

$$\sum_{p\in P}\frac{1}{p}\to\infty\,. \tag{1}$$

3 Prime Zeta Function

We need the following result about logarithms

$$\log \frac{1}{1-x} = -\log(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \quad \text{where} \quad |\,x\,| < 1\,. \tag{2}$$

Now, using the property of logarithms that

$$\log AB = \log A + \log B \tag{3}$$

and extending this for an infinite product, we get

$$\begin{aligned} \log \zeta(s) &= \sum_{p\in P} \log \frac{1}{1-p^{-s}} \\ &= \sum_{p\in P} \sum_{n=1}^{\infty} \frac{(p^{-s})^n}{n} \\ &= \sum_{p\in P} \sum_{n=1}^{\infty} \frac{p^{-sn}}{n}\,. \end{aligned} \tag{4}$$

Next, we can separate out of this summation the $n = 1$ term, leaving us with

$$\log \zeta(s) = \sum_{p\in P} p^{-s} + \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p\in P} \frac{1}{p^{sn}}\,, \tag{5}$$

where the first term on the RHS is the Euler Zeta function when summed only over the prime numbers and the second term has had the order of summation changed.

Now we have to look at convergence issues. From the second summation of the second term we have that

$$\left| \sum_{p\in P} \frac{1}{p^{sn}} \right| \le \sum_{p\in P} \frac{1}{|\,p^{sn}\,|} = \sum_{p\in P} \frac{1}{|\,p^{xn+iyn}\,|} = \sum_{p\in P} \frac{1}{p^{xn}} \le \sum_{p\in P} \frac{1}{p^n} \le \sum_{k=2}^{\infty} \frac{1}{k^n}\,, \tag{6}$$

where we have assumed that $x > 1$.

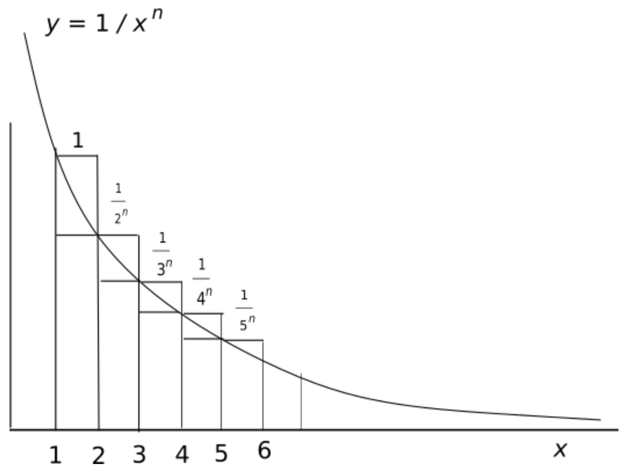

Figure 1. By moving all the rectangles, from the second one on, to the left one unit, we'll be creating a sequence of rectangles whose total area must be less than the area of the curve above it.

So, on returning to (6), we have that

$$\left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{k=2}^{\infty} \frac{1}{k^n} \leq \sum_{k=2}^{\infty} \int_{k-1}^{k} \frac{1}{t^n}\, dt = \int_1^{\infty} \frac{dt}{t^n} = \frac{1}{1-n} t^{1-n} \Big|_1^{\infty} = \frac{1}{n-1}\,, \quad (7)$$

Therefore,

$$\left| \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n} \left| \sum_{p \in P} \frac{1}{p^{sn}} \right| \leq \sum_{n=2}^{\infty} \frac{1}{n} \frac{1}{n-1} = 1\,, \qquad (8)$$

Now, we go back to (5) and take its absolute value, to get

$$|\log \zeta(s)\,| \leq \left| \sum_{p \in P} p^{-s} \right| + \left| \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}} \right| . \qquad (9)$$

Now, we know that $\lim_{s \to 1} \zeta(s) = \infty$ and thus $\lim_{s \to 1} \log \zeta(s) = \infty$ And thus $\left| \sum_{n=2}^{\infty} \frac{1}{n} \sum_{p \in P} \frac{1}{p^{sn}} \right| = 1$. Therefore,

$$\lim_{s \to 1} \sum_{p \in P} p^{-s} = \sum_{p \in P} \frac{1}{p} \to \infty\,. \qquad (10)$$