Math Diversion Problem 863

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October 25, 2025

> Even the youths shall faint and be weary, and the young men shall utterly fall: But they that wait upon the Lord shall renew their strength; they shall mount up with wings as eagles; they shall run, and not be weary; and they shall walk, and not faint.
>
> — Isaiah 40:30–31
>
> [Just a personal word of encouragement in dark times.]

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Source:   https://www.youtube.com/watch?v=3eN9tQX3JJ4
Title: Zeta Function - Part 6 - The Prime Counting Function.
  Presenter:  MrYouMath
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1 Introduction

This is the sixth part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 The Prime Counting function – Part 6

We now define the Prime Counting function $\pi(n)$ on the positive integers:

$$\pi(n) = \text{the number of primes less than or equal to } n\,, \tag{1}$$

which is one of the most important functions in all of number theory. Next, we form the difference

$$\pi(n) - \pi(n-1) = \begin{cases} 1 & \text{when } n \text{ is a prime} \\ 0 & \text{otherwise} \end{cases} \,. \tag{2}$$

We will see that this new difference function is a way to 'tag' the primes as distinct from all nonprime numbers.[1]

[1] Tagging functions like this are sometimes referred to as 'indicator functions' or 'characteristic functions'.

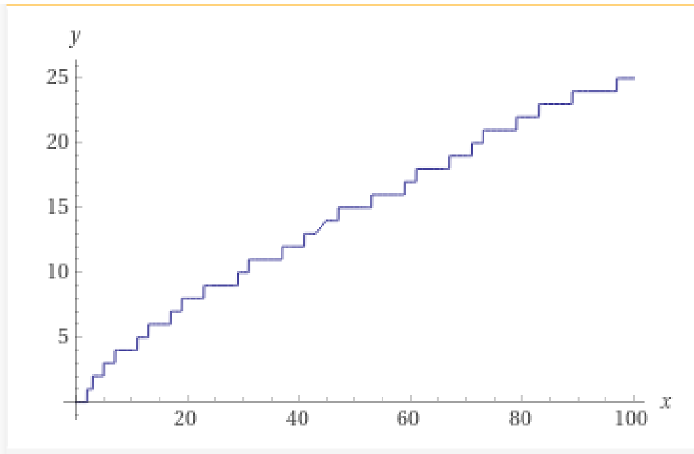

Figure 2. The Prime Counting Function. (Graphic rendered by WolframAlpha.)

We revisit one of our favorite functions and then amend it:

$$\log \zeta(s) = \sum_{p \in P} \log \frac{1}{1 - p^{-s}} = \sum_{n=2}^{\infty} (\pi(n) - \pi(n-1)) \log \frac{1}{1 - n^{-s}} \,. \tag{3}$$

So, instead of summing only on the primes we can now sum on all the integers from 2 up. The factor $(\pi(n) - \pi(n-1))$ has the effect of multiplying a prime by unity and a nonprime by zero, effectively removing that term from the summation. (This is one of the great tricks of mathematics.) Anyway, let's see what we can do with this.

$$\begin{aligned} \log \zeta(s) &= \sum_{n=2}^{\infty} (\pi(n) - \pi(n-1)) \log \frac{1}{1 - n^{-s}} \\ &= \sum_{n=2}^{\infty} \pi(n) \log \frac{1}{1 - n^{-s}} - \sum_{n=2}^{\infty} \pi(n-1) \log \frac{1}{1 - n^{-s}} \\ &= \sum_{n=2}^{\infty} \pi(n) \log \frac{1}{1 - n^{-s}} - \sum_{n=2}^{\infty} \pi(n) \log \frac{1}{1 - (n+1)^{-s}} \\ &, \end{aligned} \tag{4}$$

where, by replacing n by $n+1$ in the second term on the RHS, we lose the first term of the original sum, that being $\pi(1) \log \frac{1}{1 - 1^{-s}}$; but that term is zero anyway, since $\pi(1) = 0$. So, continuing, we can condense these two summations into just one:

$$
\begin{aligned}
\log\zeta(s) &= \sum_{n=2}^{\infty}\pi(n)\log\frac{1}{1-n^{-s}} - \sum_{n=2}^{\infty}\pi(n)\log\frac{1}{1-(n+1)^{-s}} \\
&= \sum_{n=2}^{\infty}\pi(n)\Big[\ \log\frac{1}{1-n^{-s}} - \log\frac{1}{1-(n+1)^{-s}}\ \Big] \\
&= \sum_{n=2}^{\infty}\pi(n)\Big[\ \log(1-(n+1)^{-s} - \log(1-n^{-s})\,\Big]\,, \qquad (5)
\end{aligned}
$$

In the next step, we want to replace this difference of logarithms by a definite integral. Why? Why not? Anyway, consider the following sidebar.

$$D_x \log\,(1-x^{-s}) = \frac{1}{1-x^{-s}}(sx^{-s-1}) = \frac{s}{x(x^s-1)}\,. \qquad (6)$$

Therefore, on integrating, we get

$$\log\,(1-x^{-s}) = \int \frac{s}{x(x^s-1)}\,dx\,. \qquad (7)$$

Now, continuing where we left off and employing the above tricks, we get:

$$\log\zeta(s) = \sum_{n=2}^{\infty}\pi(n)\int_n^{n+1}\frac{s}{x(x^s-1)}\,dx = \sum_{n=2}^{\infty}\int_n^{n+1}\frac{\pi(n)s}{x(x^s-1)}\,dx\,, \qquad (8)$$

where we were able to pull the $\pi(n)$ function into the integrand because it is not a function of x. Besides that, we aren't finished with using tricks – oh, no, not by a long shot. There would be no analytic number theory without tricks! We can harmlessly extend the domain of the prime counting function from the positive integers to the positive reals, and get

$$\log\zeta(s) = \sum_{n=2}^{\infty}\int_n^{n+1}\frac{\pi(x)s}{x(x^s-1)}\,dx\,. \qquad (9)$$

To finish off this section, we need one more trick, but only a small one. The integral of a function from n to $n+1$ is the area under the curve between the integers n and $n+1$. But then we add up all these area strips by summing from 2 to ∞. This means that we can replace the combination of summation and integral with just an integral

$$\log\zeta(s) = \int_2^{\infty}\frac{s\pi(x)}{x(x^s-1)}\,dx\,. \qquad (10)$$