Math Diversion Problem 865

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I was trying to reduce the number of string theories by showing
that some were inconsistent, but what happened instead
was to reduce the number of string theories by showing
that they were different limits of each other.
— Edward Witten
[*Closer to the Truth* (13 Mar 2021)]

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Source:   https://www.youtube.com/watch?v=GeKDmoYHiAk
Title: Zeta Function - Part 7: $\zeta(2)$ is calculated.
  Presenter:  MrYouMath
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1 Introduction

This is the seventh part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 $\zeta(2)$ calculated

Well, we finally get around to something 'practical'. But to appreciate this, we need another trick. You remember that the Fundamental Theorem of Algebra is that every polynomial over the complex numbers has a root in the complex numbers. By extension, every polynomial of order n has n roots and thus can be factored into n linear factors.

So, for example, polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + d$ has four roots, say, r_1, r_2, r_3, r_4. Therefore

$$f(x) = (x - r_1)(x - r_2)(x - r_3)(x - r_4)\,. \tag{1}$$

But why does this work? Simple. By definition, a root r of a function $g(x)$ is a value on which the function becomes zero, or $g(r) = 0$. Okay. Look at (1). If we take $f(r_1)$, we get

$$f(r_1) = (r_1 - r_1)(r_1 - r_2)(r_1 - r_3)(r_1 - r_4) = 0\,, \tag{2}$$

and likewise for all the other roots.

Okay, so this trick works for any polynomial, but a polynomial has only finite order. That means that a polynomial has a highest power term. Therefore polynomials do not have a sum of progressively higher power to infinity. But that fact is not going to stop us from reasoning analogously

So, we begin with the power series expansion of $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots , \tag{3}$$

Now we look at a particular argument of this sine function, that being πs and then divide through by πs:

$$\frac{\sin \pi s}{\pi s} = 1 - \frac{(\pi s)^2}{3!} + \frac{(\pi s)^4}{5!} - \frac{(\pi s)^6}{7!} + \cdots , \tag{4}$$

The zeros of this function are obviously all the integers except 0, which is ruled out by the denominator. So, let's just rewrite this function as a product of all its zeros, as if we know what we're doing. To deal with $s = \pm 1$, we get

$$\frac{\sin \pi s}{\pi s} = \Big(1 + \frac{s}{1}\Big)\Big(1 + \frac{-s}{1}\Big) \cdots . \tag{5}$$

For $s = \pm 2$, we expand this to

$$\frac{\sin \pi s}{\pi s} = \Big(1 + \frac{s}{1}\Big)\Big(1 + \frac{-s}{1}\Big)\Big(1 + \frac{s}{2}\Big)(1 + \frac{-s}{2}\Big) \cdots . \tag{6}$$

To include $s = \pm 3$, we expand this to

$$\frac{\sin \pi s}{\pi s} = \Big(1 + \frac{s}{1}\Big)\Big(1 + \frac{-s}{1}\Big)\Big(1 + \frac{s}{2}\Big)\Big(1 + \frac{-s}{2}\Big)\Big(1 + \frac{s}{3}\Big)\Big(1 + \frac{-s}{3}\Big) \cdots , \tag{7}$$

and so forth to infinity. But we can simplify this down to

$$\frac{\sin \pi s}{\pi s} = \Big(1 - \frac{s^2}{1^2}\Big)\Big(1 - \frac{s^2}{2^2}\Big)\Big(1 - \frac{s^2}{3^2}\Big) \cdots . \tag{8}$$

Next, we expand the RHS of this in powers of s^2, to get

$$\frac{\sin \pi s}{\pi s} = 1 - \Big(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\Big)s^2 + \Big(\frac{1}{1^2 2^2} + \frac{1}{1^2 3^2} + \frac{1}{2^2 3^2} + \cdots\Big)s^4 - \Big(\frac{1}{1^2 2^2 3^2} + \cdots\Big)s^6 \cdots . \tag{9}$$

Now, if we compare the quadratic term of this last equation with the quadratic term of (4), we get

$$-\frac{\pi^2}{3!} = -\Big(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\Big) . \tag{10}$$

And now we are ready to write down $\zeta(2)$

$$\zeta(2) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6} . \tag{11}$$