Math Diversion Problem 867

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In life, work hard, don't complain, emphasize the
positive, earn your accomplishments.
— The Author

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Source:   https://www.youtube.com/watch?v=axQqExF7NsU (1st)
  https://www.youtube.com/watch?v=XHQOOzqTjd0 (2nd)
  https://www.youtube.com/watch?v=1f24RZfP6m8 (3rd)
Title: Zeta Function - Part 8:$\zeta(2n)$ and the
             Bernoulli Numbers
  Presenter:  MrYouMath
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1 Introduction

This is the eighth part of a 14-part series on the Zeta function. What I'm presenting here is what I refer to as the 'read-a-long notes' to the videos. They are brief on explanations. For better explanations, please see the videos by MrYouMath, as listed above.

2 $\zeta(2n)$ and the Bernoulli Numbers – Part 8

Let's return to

$$\frac{\sin \pi s}{\pi s} = \prod_{k=1}^{\infty} \left(1 - \frac{s^2}{k^2}\right), \tag{1}$$

and multiply through by πs and then take the logarithm:

$$\log \sin \pi s = \log \pi s + \sum_{k=1}^{\infty} \log \left(1 - \frac{s^2}{k^2}\right). \tag{2}$$

Now we differentiate by s and then multiply through by s:

$$\pi s \cot \pi s = 1 + 2s^2 \sum_{k=1}^{\infty} \frac{1}{s^2 - k^2}. \tag{3}$$

If we take this last equation and divide through by s and reorganize it, we get

$$\pi \cot \pi s = \frac{1}{s} + \sum_{k=1}^{\infty} \frac{2s}{s^2 - k^2} = \frac{1}{s} + \sum_{k=1}^{\infty} \Big(\frac{1}{s-k} + \frac{1}{s+k} \Big) = \sum_{k=-\infty}^{\infty} \frac{1}{s+k} \,. \tag{4}$$

Let's return to Eq. (3) and put it into this alternative form:

$$\pi s \cot \pi s = 1 + \sum_{k=1}^{\infty} \frac{1}{\big(1 - s^2/k^2\big)} \big(- 2s^2/k^2 \big) \,. \tag{5}$$

Now we replace the factor $\frac{1}{\left(1-s^2/k^2\right)}$ by its equivalence as a geometric series

$$\begin{aligned} \pi s \cot \pi s &= 1 + \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \Big(\frac{s^2}{k^2} \Big)^n \big(- 2s^2/k^2 \big) \\ &= 1 - 2 \sum_{k=1}^{\infty} \sum_{n=0}^{\infty} \Big(\frac{s^2}{k^2} \Big)^{n+1} \\ &= 1 - 2 \sum_{n=1}^{\infty} \Big[\sum_{k=1}^{\infty} \frac{1}{k^{2n}} \Big] s^{2n} \\ &= 1 - 2 \sum_{n=1}^{\infty} \zeta(2n) s^{2n} \,. \end{aligned} \tag{6}$$

Therefore, we have the result

$$\pi s \cot \pi s = 1 - 2 \sum_{n=1}^{\infty} \zeta(2n) s^{2n} \,. \tag{7}$$

Introducing the Bernoulli Numbers

The reason we care about the Bernoulli numbers in this context is because with them we are able to write a formula for the ζ function in terms of them. But to get there, we need some preliminary results.

Remembering that $\cot x = \cos x / \sin x$, we write in terms of their complex counterparts

$$\begin{aligned} \pi s \cot \pi s &= \pi s \frac{e^{i\pi s} + e^{-i\pi s}}{2} \cdot \frac{2}{e^{i\pi s} - e^{-i\pi s}} \\ &= \pi s i \left(1 + \frac{2}{e^{2i\pi s} - 1} \right) \,. \end{aligned} \tag{8}$$

Next, we define the Bernoulli numbers β_n as those numbers that satisfy[1]

$$\frac{s}{e^s - 1} = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} s^n \,. \tag{9}$$

[1] A full treatment of the Bernoulli numbers would answer the question of why the Bernoulli numbers are defined as they are, but that is beyond the scope of this presentation.

Moving on. Since

$$e^s = \sum_{n=0}^{\infty} \frac{s^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{s^n}{n!} \,, \tag{10}$$

then

$$\frac{s}{\sum_{n=1}^{\infty} \frac{s^n}{n!}} = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} s^m \,, \tag{11}$$

where we changed the index of summation on the RHS so we don't get confused. On rewriting, we get

$$1 = \sum_{m=0}^{\infty} \frac{\beta_m}{m!} s^m \sum_{n=1}^{\infty} \frac{s^{n-1}}{n!} \,, \tag{12}$$

Next, we employ the Cauchy Product Formula to convert one of the infinite summations to a finite summation.

$$1 = \sum_{n=0}^{\infty} \sum_{\mu=0}^{n} \frac{\beta_\mu}{\mu!} \frac{1}{(n-\mu+1)!} s^n \,. \tag{13}$$

Now, if we multiply and divide by $(n+1)!$, we have that

$$\begin{aligned} 1 &= \sum_{n=0}^{\infty} \sum_{\mu=0}^{n} \frac{1}{(n+1)!} \beta_\mu \frac{(n+1)!}{\mu!(n-\mu+1)!} s^n \\ &= \sum_{n=0}^{\infty} \sum_{\mu=0}^{n} \frac{1}{(n+1)!} \beta_\mu \binom{n+1}{\mu} s^n \\ &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left[\sum_{\mu=0}^{n} \binom{n+1}{\mu} \beta_\mu \right] s^n \,. \end{aligned} \tag{14}$$

Obviously, by comparing the LHS to the RHS of this equation the only term on the right that isn't identically zero is the constant term. So, after accounting for the constant term, we get that $\beta_0 = 1$. So, all the higher-order terms in powers of s are zero, therefore

$$\sum_{\mu=0}^{n} \binom{n+1}{\mu} \beta_\mu = 0 \,. \tag{15}$$

Let's work it out some some small values of n. For $n = 1$:

$$\binom{2}{0} \beta_0 + \binom{2}{1} \beta_1 = 0 \,. \tag{16}$$

Solving this for β_1:

$$\beta_1 = -\tfrac{1}{2} \beta_0 = -\tfrac{1}{2} \,. \tag{17}$$

One more. For $n=2$:

$$\binom{3}{0}\beta_0 + \binom{3}{1}\beta_1 + \binom{3}{2}\beta_2 = 0\,. \tag{18}$$

Solving this for β_2:

$$\beta_2 = -\tfrac{1}{3}[\,1 + 3(-\tfrac{1}{2})\,] = \tfrac{1}{6}\,. \tag{19}$$

The odd values of n for $n>1$ all vanish. And for furture needs: $\beta_4 = -1/30$.

Now we return to (8):

$$\pi s \cot \pi s = \pi s i + \frac{2\pi s i}{e^{2i\pi s} - 1}\,. \tag{20}$$

And, (9) and letting $s \to \pi s i$, we have that

$$\begin{aligned}\pi s \cot \pi s &= \pi s i + \sum_{n=0}^{\infty} \frac{\beta_n}{n!}(2\pi s i)^n \\ &= \pi s i + \frac{\beta_0}{0!} + \frac{\beta_1}{1!}(2\pi s i) - 2\sum_{n=2}^{\infty}\Big(-\frac{1}{2}\Big)\frac{\beta_n}{n!}(2i\pi s)^n \\ &= 1 - 2\sum_{n=2}^{\infty}\Big(-\frac{1}{2}\Big)\frac{\beta_n}{n!}(2\pi i s)^n\,. \end{aligned} \tag{21}$$

Now, since all the β_n's are zero for odd $n>1$, then we can continue

$$\begin{aligned}\pi s \cot \pi s &= 1 - 2\sum_{n=1}^{\infty}\frac{\beta_{2n}}{(2n)!}\Big(-\frac{1}{2}\Big)(2\pi i s)^{2n} \\ &= 1 - 2\sum_{n=1}^{\infty}\frac{\beta_{2n}}{(2n)!}(-1)\frac{(2\pi)^{2n} i^{2n}}{2} s^{2n} \\ &= 1 - 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}(2\pi)^{2n}\beta_{2n}}{2(2n)!} s^{2n}\,, \end{aligned} \tag{22}$$

where we used that $i^{2n} = (i^2)^n = (-1)^n$.

Therefore, on comparing this last equation to (7), we have that

$$\zeta(2n) = (-1)^{n+1}\frac{(2\pi)^{2n}\beta_{2n}}{2(2n)!}\,. \tag{23}$$

And we can test this formula for small values of n: For $n=1$, we have

$$\zeta(2) = (-1)^1\frac{(2\pi)^2\beta_2}{2(2)!} = \frac{\pi^2}{6}\,. \tag{24}$$

For $n=2$, we have

$$\zeta(4) = (-1)^3\frac{(2\pi)^4\beta_4}{2(4)!} = -\frac{1}{3}\pi^4\Big(-\frac{1}{30}\Big) = \frac{\pi^4}{90}\,. \tag{25}$$