Math Diversion Problem 868

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A clue is anything that doesn't happen
the way it oughtta happen.
— Harry Orwell, TV
show *Harry O*

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Source: https://www.youtube.com/watch?v=NGYX6gtObec
Title:  Introduction to conformal field theory, Lecture 1
 Presenter: Tobias Osborne
 (Read-along notes and a problem to solve.)
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1 Introduction

Previous posts on Conformal Field Theory by Tobias Osborne are to be found at Problems 775, 780, and 786.

2 Problem

We have been working with the following relation from conformal field theory for some time now:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} (\partial \cdot \epsilon) \, \eta_{\mu\nu} \,, \tag{1}$$

By restricting the underlying vector space to $\mathbb{R}^{p,q} = \mathbb{R}^{2,0}$, show that the resulting coordinates can be interpreted as satisfying the Cauchy-Riemann equation.

3 Solution

We begin with

$$d = p + q = 2 + 0 = 2 \,. \tag{2}$$

So that (1) becomes

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = (\partial \cdot \epsilon) \, \eta_{\mu\nu} = (\partial \cdot \epsilon) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \,, \tag{3}$$

where we have lowered the indices as we are in Euclidean space. Next, we note that

$$\partial \cdot \epsilon = \partial_1 \epsilon_1 + \partial_2 \epsilon_2 \,, \tag{4}$$

So, using (4) in (1) and exploring the four combined values of μ and ν, we have that

$$(\mu = 1,\ \nu = 1) \quad \partial_1 \epsilon_1 + \partial_1 \epsilon_1 = \partial_1 \epsilon_1 + \partial_2 \epsilon_2 \,, \tag{5a}$$
$$(\mu = 1,\ \nu = 2) \quad \partial_1 \epsilon_2 + \partial_2 \epsilon_1 = 0 \,, \tag{5b}$$
$$(\mu = 2,\ \nu = 1) \quad \partial_2 \epsilon_1 + \partial_2 \epsilon_1 = 0 \,, \tag{5c}$$
$$(\mu = 2,\ \nu = 2) \quad \partial_2 \epsilon_2 + \partial_2 \epsilon_2 = \partial_1 \epsilon_1 + \partial_2 \epsilon_2 \,. \tag{5d}$$

Form (5a) and (5d) , we get

$$\partial_1 \epsilon_1 = \partial_2 \epsilon_2 \,, \tag{6}$$

and from either (5b) or (5c), we get

$$\partial_1 \epsilon_2 = -\partial_2 \epsilon_1 \,, \tag{7}$$

A Reinterpretation:

Let

$$\boldsymbol{\epsilon} = \epsilon_1 + i \epsilon_2 \,, \tag{8}$$

and let

$$z = x^1 + i x^2 \,,$$
$$\bar{z} = x^1 - i x^2 \,. \tag{9}$$

Then,

$$\boldsymbol{\epsilon}(z) = \epsilon^1 + i \epsilon^2 \quad \text{and} \quad \bar{\boldsymbol{\epsilon}}(\bar{z}) = \epsilon^1 - i \epsilon^2 \,, \tag{10}$$

then, since $\boldsymbol{\epsilon}$ satisfies the Cauchy-Riemann equations then $\boldsymbol{\epsilon}$ is analytic on an infinitesimal domain.